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Pointwise bounds in parabolic and elliptic partial differential equations.

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POINTWISE BOUNDS IN PARABOLIC AND ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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[illegible]

APPROVAL SHEET

Title of Thesis: Pointwise Bounds in Parabolic and Elliptic Partial Differential Equations

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ABSTRACT

Title of Thesis: Pointwise Bounds in Parabolic and Elliptic Partial Differential Equations

Lieutenant Fred James Bellar, Jr., United States Navy, Doctor of Philosophy, 1961

Thesis directed by: L. E. Payne, Research Professor, Institute for Fluid Dynamics and Applied Mathematics.

A method is presented for obtaining explicit upper and lower pointwise bounds for the solutions of rather general interior boundary value problems. The differential equations associated with these problems are of the elliptic type in certain sections of the thesis while both linear and non-linear parabolic problems are the subject of investigation in other sections. The bounds are in terms of the integrals of the squares of known functions and hence, in the linear case, improvement is possible using the Rayleigh-Ritz technique.

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POINTWISE BOUNDS IN PARABOLIC AND ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

by

Fred James Bellar, Jr.

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CHAPTER I

INTRODUCTION

In [21] Payne and Weinbeger presented a method for determining pointwise bounds for the solution of Dirichlet and mixed type boundary value problems for arbitrary second order elliptic equations and rather general domains. In the present paper the writer will obtain similar results in the parabolic case for Dirichlet and mixed type boundary value problems where the given data is a function of the domain for some problems while for others the data may depend on both the solution and the domain of the problem.

The methods of this paper yield error estimates in terms of the integral of the square of known functions and thus the results lend themselves to improvement by means of the Rayleigh-Ritz technique when the given data is a function of domain (the linear case). In such cases it is an unfortunate fact that, even if the set of functions to which one applies the Rayleigh-Ritz technique is complete in the function space of the solution, the actual solution of the problem can not be obtained using our present-day computers due to the machine creation of large round off errors. It is quite possible however that future generations of these computing devices will eliminate

such errors and thus make the techniques of this paper of greater practical importance. In this regard we note that the current electronic computing machines have made of practical interest problem solving techniques which were previously only of theoretical interest since feasible uses were not apparent at the time of their evolution.

For many of the problems considered in this thesis, the existence of a solution has not been established under the general conditions necessary for the existence of the pointwise bound. When certain addition requirements or modifications are placed on the boundary value problem, by referring to the literature [1], [5], [9] to [15], [22] and [23] we note that the existence of the solution is guaranteed.

As was pointed out above, the estimates of this paper are in terms of the integrals of known functions. For another method of estimating the solution function of boundary value problems, the reader is referred to the excellent works of Fichera [6] to [8] where the solution function is bounded in terms of the maximum norm. Using the results of Fichera as a starting point, Lieberstein [19] derived a numerical procedure by which error bounds were obtained using a digital computer.

In Chapter II we obtain some preliminary results which shall form the basis of many of the conclusions of the following chapters. The identity of Hörmander [16],

[illegible]

which was used by Payne and Weinberger [21], is modified in order to allow its use in a Dirichlet type parabolic problem and functions analogous to (4.1) and (4.18) of [21] are introduced and investigated. Chapter III concerns itself with non-linear normally parabolic problems of both the Dirichlet and mixed types. For the purposes of this thesis a normally parabolic differential operator has the form

$$J(w) = \sum_{ij} \frac{\partial}{\partial x^i}\left(a^{ij} \frac{\partial w}{\partial x^j}\right) - z \frac{\partial w}{\partial y} \tag{1.1}$$

where the matrix a^{ij} is positive definite and z is strictly positive. We shall see that the desired estimates are obtained by multiplying the differential operator by a parametrix function and then using Green's theorem to generate the solution at the point in question. The unknown terms in the resulting expression are estimated as a consequence of the results of Chapter II and by further use of the divergence theorem. In Chapter IV the problems considered are degenerate in character, thus the matrix a^{ij} becomes positive semi-definite for some problems while the function z need only be non-negative for others. The next to the last chapter of the paper outlines a method by which solution bounds may be generated for a class of elliptic problems which were not considered by Payne and Weinberger in their investigation of such problems [21]. In Chapter VI the results of the thesis are applied to obtain bounds for the solution of a specific boundary value problem.

CHAPTER II

PRELIMINARY RESULTS

1. General Definitions

Let V be an open $(N + 1)$ dimensional domain of real variables $(x, y) = (x^1, x^2 \ldots x^N, y)$ which is bounded by two hyperplanes, $y = o$ and $y = y_o > o$, and a surface S lying between these planes. As usual we denote the closure of V by $\overline{V}$ and assume that the divergence theorem holds in $\overline{V}$. The reader is referred to Chapter IV of [18] for information relating to the most general regions in which this theorem is valid. Let

2.1 $$D(\eta) = \left\{(x,y) \middle| \; y = \eta, (x,y) \in \overline{V} - \overline{S}\right\}$$

then

2.2 $$\overline{V} = V \cup \overline{D}(o) \cup \overline{D}(y_o) \cup S = V \cup D(o) \cup D(y_o) \cup \overline{S}.$$

On the boundary of V the Euclidean outward normal is denoted by

2.3 $$(n_i, n_y) = (n_1, n_2 \ldots n_N, n_y)$$

where

2.4 $$n_y^2 + \sum_1^N n_i^2 = 1 .$$

Since $\overline{V}$ is a region in which the divergence theorem is valid, it is clear that 2.3 is (uniquely) defined almost everywhere on the boundary of V. In addition, unless otherwise indicated, we require that S be such that

2.5 $$\min_{S} \sum_{1}^{N} n_i^2 = m_1 > o$$

for all $(x,y) \in S$ at which the normal vector is defined.

A function $Z = Z(x,y)$ is said to be piecewise continuous in $\overline{V}$ if $\overline{V}$ may be divided into a finite number of subregions in each of which the function is continuous and has a finite limit as the boundary is approached. The above is the usual definition of piecewise continuity; the following non-standard definitions shall also be employed since they greatly simplify the presentation of the problem. When a function $Z(x,y)$ is said to be continuously (piecewise continuously) differentiable in x for $(x,y) \in V$ then Z is continuous on D(y) for $o < y < y_o$ and is piecewise continuous in V while the $\frac{\partial Z}{\partial x^i}$, $i = 1, 2 \ldots N$, are continuous (piecewise continuous) on D(y) for $o < y < y_o$ and piecewise continuous in V. Thus if Z is twice continuously differentiable in x and piecewise continuously differentiable in y for $(x,y) \in V$ then Z is continuous in V, the $\frac{\partial Z}{\partial x^i}$, $i = 1, 2 \ldots N$ are continuous on D(y) for $o < y < y_o$ and piecewise continuous in V, the $\frac{\partial^2 Z}{\partial x^i \partial x^j}$, $ij = 1, 2 \ldots N$, are continuous on D(y) for $o < y < y_o$ and piecewise continuous in V, and finally $\frac{\partial Z}{\partial y}$

is piecewise continuous over the entire volume.

2. <u>The Parametrix of the Parabolic Problem</u>

For $(x,y) \in V$ let the following differential operator be defined

$$2.6 \qquad J(\psi) = (a^{ij}\psi_{,j})_{,i} + \frac{\partial}{\partial y}(Z\psi)$$

where the symbol $_{,i}$ indicates partial differentiation with respect to x^i, the summation convention is used throughout and the components of the symmetric matrix $a^{ij} = a^{ij}(x,y)$ are piecewise continuously differentiable in x for $(x,y) \in V$ and are such that

$$2.7 \qquad o \leqslant a_o \sum_1^N \xi_i^2 \leq a^{ij}\xi_i \xi_j \leq a_1 \sum_1^N \xi_i^2$$

for any numbere $(\xi_1, \ldots \xi_N)$ and all $(x,y) \in \overline{V}$ with equality on the left holding if and only if $\xi_i = o$ for i=1, 2 . . . N. The function $Z = Z(x,y)$ is piecewise continuously differentiable in y for $(x,y) \in \overline{V}$. For the point $p = (x_o, y_o) \in D(y_o)$, at which a bound for the solution of certain boundary value problems shall later be obtained, we assume the functions a^{ij} and Z satisfy a Lipschitz condition for $(x,y) \in \overline{V}$; that is, we suppose there exist positive numbers A^{ij} and A such that

$$2.8 \qquad \left| a^{ij}(x,y) - a^{ij}(p) \right| \left\{ (y_o - y)^2 + \sum_1^N (x^i - x_o^i)^2 \right\}^{-\frac{1}{2}} \leqslant A^{ij}$$

$$2.9 \qquad \left| Z(x,y) - Z(p) \right| \left\{ (y_o - y)^2 + \sum_1^N (x^i - x_o^i) \right\}^{-\frac{1}{2}} \leqslant A$$

for $(x,y)\in\bar{V}$.

The parametrix function γ_p is defined with respect to the point p by

2.10 $$\gamma_p = \left[4\pi(y_o - y)\right]^{-\frac{N}{2}} Z(p)^{\frac{N-2}{2}} \sqrt{a(p)} \exp\left[-z(p)\rho^2 \langle 4(y_o - y)\rangle^{-1}\right]$$

for

2.11 $$p = (x_o, y_o) \in D(y_o), \ (x,y)\in V, \text{ and } z(p) > o$$

where

2.12 $$\rho^2 = a_{rs}(p)(x^r - x_o^r)(x^s - x_o^s)$$

and a(p) is the determinant of the inverse matrix $a_{ij}(p)$. On the surface $D(y_o) - \{p\}$, the parametrix and its derivatives are defined as the limits of these functions as $(x,y) \to (x, y_o)$ with $(x,y)\in V$ and $x \neq x_o$.

<u>Theorem</u> <u>2.1</u>. The parametrix γ_p has the following properties:

a) At all points $(x,y)\in\bar{V} - \{p\}$, the parametrix is twice continuously differentiable.

b) The function

2.13 $$(y_o - y)^{\frac{N}{2}+\lambda} \exp\left[\varsigma (y_o - y)^{-1} z(p)\rho^2\right] \bar{J}(\gamma_p)^2$$

is integrable over V for ς and λ satisfying

2.14 $$\varsigma < \tfrac{1}{2} \qquad \text{and} \qquad o < \lambda .$$

c) For any bounded function Ψ which is continuous on $V \cup D(y_0)$ the parametrix is such that

2.15 $$\lim_{\substack{y \to y_0 \\ y < y_0}} \iint_{D(y)} \Psi Z \gamma_p \, ds = \Psi(p) = \Psi(x_0, y_0).$$

Proof: From the definition of γ_p, it is obvious that part (a) is satisfied. To establish part (b) of the theorem we note that for c_i, $i = 2, 3 \ldots$ appropriately defined (the constant c_1 is reserved for later use)

2.16 $$\iiint_V (y_0-y)^{\frac{N}{2}+\lambda} \exp\left[\varsigma (y_0-y)^{-1} z(p)\rho^2\right] \mathcal{F}(\gamma_p)^2 \, dV$$

$$\leq c_2 \iiint_V (y_0-y)^{-\frac{N}{2}+\lambda} \Big\{ + z(p)^2 \left\langle a^{ij}_{,i} a_{kj}(p)(x^k - x_0^k)\right\rangle^2 \left\langle 4(y_0-y)^2\right\rangle^{-1}$$

$$+ \left(\frac{\partial z}{\partial y}\right)^2 + \left\langle -z(p) a^{ij} a_{ij}(p) + Nz\right\rangle^2 \left\langle 4(y_0-y)^2\right\rangle^{-1} + \Big\langle z(p)^2 a^{ij} a_{kj}(p)(x^k - x_0^k) \cdot$$

$$\cdot a_{i\ell}(p)(x^\ell - x_0^\ell) - z z(p)\rho^2\Big\rangle^2 \left\langle 4(y_0-y)^2\right\rangle^{-2} \Big\} \exp\left[-(\tfrac{1}{2} - \varsigma) z(p)\rho^2 (y_0-y)^{-1}\right] dV$$

$$\leq c_3 + \int_{y_0-\delta}^{y_0} \int_{\omega_N} \int_0^{\varepsilon} \Big\{ c_4 r^{N+1} (y_0-y)^{-\frac{N}{2}-2+\lambda} + c_5 r^{N-1} (y_0-y)^{-\frac{N}{2}+\lambda}$$

$$+ c_2 (y_0-y)^{-\frac{N}{2}+\lambda} \Big[\Big\langle -z(p) g^{ij}_{k} a_{ij}(p)(x^k - x_0^k) - z(p) g^{ij}_{y} a_{ij}(p)(y_0-y)$$

$$+ N h_i (x^i - x_0^i) + N h_y (y_0-y)\Big\rangle^2 \left\langle 4(y_0-y)^2\right\rangle^{-1} + \Big\langle z(p)^2 g^{ij}_{r} a_{kj}(p)(x^k - x_0^k) a_{i\ell}(p)(x^\ell - x_0^\ell) \cdot$$

$$\cdot (x^r - x_0^r) + z(p)^2 g^{ij}_{y} a_{kj}(p)(x^k - x_0^k) a_{i\ell}(p)(x^\ell - x_0^\ell)(y_0-y) - h_i z(p)\rho^2 (x^i - x_0^i)$$

$$-h_y z(p)\rho^2 (y_0-y)^2 \langle 4(y_0-y)^2\rangle^{-2}\Big] \exp\left[-\left(\tfrac{1}{2}-\varsigma\right) z(p) r^2 (a_1 y_0 - a_1 y)^{-1}\right] dr\, d\omega\, dy$$

$$\leq C_3 + \int_{y_0-\delta}^{y_0} \int_{\omega_N} \int_0^{\varepsilon} \Big\{ C_6 r^{N+1} (y_0-y)^{-\frac{N}{2}-2+\lambda} + C_7 r^{N-1} (y_0-y)^{-\frac{N}{2}+\lambda}$$

$$+ C_8 r^{N+5} (y_0-y)^{-\frac{N}{2}-4+\lambda} + C_9 r^{N+3} (y_0-y)^{-\frac{N}{2}-2+\lambda} \Big\} \cdot$$

$$\cdot \exp\left[-\left(\tfrac{1}{2}-\varsigma\right) z(p)\, r^2 \langle a_1 (y_0-y)\rangle^{-1}\right] dr\, d\omega\, dy$$

where

2.17 $$r^2 = \sum_1^N (x^i - x_0^i)^2$$

2.18 $$g_k^{ij}(x^k - x_0^k) + g_y^{ij}(y_0 - y) = a^{ij}(x,y) - a^{ij}(p) \quad i,j = 1,2 \ldots N$$

with

2.19 $$\begin{cases} g_k^{ij} = (x^k - x_0^k)\left[(y_0-y)^2 + \sum_1^N (x^l - x_0^l)^2\right]^{-1} \left\{a^{ij}(x,y) - a^{ij}(p)\right\} & i,j,k = 1, \ldots, N \\ g_y^{ij} = (y_0-y)\left[(y_0-y)^2 + \sum_1^N (x^l - x_0^l)^2\right]^{-1} \left\{a^{ij}(x,y) - a^{ij}(p)\right\} & i,j = 1, \ldots, N \end{cases}$$

and

2.20 $$h_i(x^i - x_0^i) + h_y(y_0 - y) = z(x,y) - z(p)$$

with

2.21 $$\begin{cases} h_i = (x^i - x_0^i)\left[(y_0-y)^2 + \sum_1^N (x^l - x_0^l)^2\right]^{-1} \left\{z(x,y) - z(p)\right\} & i = 1, \ldots, N \\ h_y = (y_0-y)\left[(y_0-y)^2 + \sum_1^N (x^l - x_0^l)^2\right]^{-1} \left\{z(x,y) - z(p)\right\}. \end{cases}$$

Now since the a^{ij} and z satisfy 2.8 and 2.9, the functions g_k^{ij}, g_y^{ij}, h_i and h_y are bounded over V and thus the constants on the right of 2.16 exist. By integration by parts with respect to r and then y, it is easy to establish (using 2.14) that each of the integrals on the right of 2.16 is bounded and thus part (b) of the theorem is proved.

To establish the validity of 2.15 we suppose that $\varepsilon_0 > 0$ is chosen so that for $0 < \varepsilon < \varepsilon_0$ we have

2.22
$$\lim_{\substack{y \to y_0 \\ (x,y) \in V}} \iint_{D(y)} \psi z \gamma_p \, ds = \lim_{\substack{y \to y_0 \\ (x,y) \in V}} \int_{\omega_N} \int_0^{\varepsilon} \psi z \gamma_p \, ds$$

$$+ \lim_{\substack{y \to y_0 \\ (x,y) \in V}} \iint_{D(y) - \left\{(x,y) \mid \varepsilon^2 \geq \sum_1^N (x^i - x_0^i)^2\right\}} \psi z \gamma_p \, ds$$

$$= \lim_{\substack{y \to y_0 \\ (x,y) \in V}} \int_{\omega_N} \int_0^{\varepsilon} \psi z \gamma_p \, ds .$$

Since 2.22 is true for every ε such that $0 < \varepsilon < \varepsilon_0$ and since z and ψ are continuous at $p \in D(y_0)$, we have that

2.23
$$\lim_{\substack{y \to y_0 \\ (x,y) \in V}} \iint_{D(y)} \psi z \gamma_p \, ds$$

$$= \lim_{\varepsilon \to 0} \lim_{\substack{y \to y_0 \\ (x,y) \in V}} \int_{\omega_N} \int_0^{\varepsilon} \psi z \gamma_p \, ds$$

$$= \lim_{\varepsilon\to 0}\lim_{y\to y_0}\left\{M(\varepsilon,y)\int_{\omega_N}\int_0^{\varepsilon}\left[4\pi(y_0-y)\right]Z(p)^{\frac{N-2}{2}} r^{N-1}\exp\left[-Z(p)r^2\langle 4(y_0-y)\rangle^{-1}\right]dr\,d\omega\right\}$$

$$\min_{\{(x,y)\mid\sum_1^N(x^i-x_0^i)^2\le\varepsilon^2\}}\Psi Z \le M(\varepsilon,y)\le \max_{\{(x,y)\mid\sum_1^N(x^i-x_0^i)^2\le\varepsilon^2\}}\Psi Z$$

$$= \Psi(p)Z(p)^{\frac{N}{2}}(4\pi)^{-\frac{N}{2}}\lim_{\varepsilon\to 0}\lim_{y\to y_0}\int_{\omega_N}\int_0^{\varepsilon}(y_0-y)^{-\frac{N}{2}}r^{N-1}\exp\left[-Z(p)r^2\langle 4(y_0-y)\rangle^{-1}\right]dr\,d\omega$$

$$= \Psi(p)Z(p)^{\frac{N}{2}-1}(4\pi)^{-\frac{N}{2}}\left\{\lim_{\varepsilon\to 0}\lim_{y\to y_0}\int_{\omega_N}(-2)\varepsilon^{N-2}(y_0-y)^{-\frac{N}{2}+1}\exp\left[-Z(p)\varepsilon^2\langle 4(y_0-y)\rangle^{-1}\right]d\omega\right.$$

$$+\lim_{\varepsilon\to 0}\lim_{y\to y_0}\int_{\omega_N}0\cdot(y_0-y)^{-\frac{N}{2}+1}d\omega$$

$$+\lim_{\varepsilon\to 0}\lim_{y\to y_0}\int_{\omega_N}\int_0^{\varepsilon}2(N-2)r^{N-3}(y_0-y)^{-\frac{N}{2}+1}\exp\left[-Z(p)r^2\langle 4(y_0-y)\rangle^{-1}\right]dr\,d\omega$$

$$= \Psi(p)Z(p)^{\frac{N}{2}-1}(4\pi)^{-\frac{N}{2}}\left\{\lim_{\varepsilon\to 0}\lim_{y\to y_0}\int_{\omega_N}\int_0^{\varepsilon}2(N-2)\,r^{N-3}(y_0-y)^{-\frac{N}{2}+1}\cdot\right.$$

$$\left.\cdot\exp\left[-Z(p)r^2\langle 4(y_0-y)\rangle^{-1}\right]dr\,d\omega\right\}$$

$$= \Psi(p)Z(p)^{\frac{N}{2}-2}(4\pi)^{-\frac{N}{2}}\left\{\lim_{\varepsilon\to 0}\lim_{y\to y_0}\int_{\omega_N}\int_0^{\varepsilon}2^2\cdot 2^2\left(\frac{N-2}{2}\right)\left(\frac{N-4}{2}\right)r^{N-5}\cdot\right.$$

$$\left.\cdot(y_0-y)^{-\frac{N}{2}+2}\exp\left[-Z(p)r^2\langle 4(y_0-y)\rangle^{-1}\right]dr\,d\omega\right\}.$$

Hence if N is even we obtain

2.24 $$\lim_{y\to y_0}\iint_{D(y)}\Psi Z\gamma_p\,ds =$$

$$= \psi(p)\, z(p)\,(4\pi)^{-\frac{N}{2}} \lim_{\varepsilon\to 0}\lim_{y\to y_0}\int_{\omega_N}\int_0^{\varepsilon} 4^{\frac{N-2}{2}}\left(\frac{N-2}{2}\right)!\, r\,(y_0-y)^{-1}\cdot$$

$$\cdot \exp\left[-z(p)\, r^2\langle 4(y_0-y)\rangle^{-1}\right] dr\, d\omega$$

$$= -\psi(p)\,(4\pi)^{-\frac{N}{2}} \lim_{\varepsilon\to 0}\lim_{y\to y_0}\int_{\omega_N} 4^{\frac{N-1}{2}}\left(\frac{N-2}{2}\right)! \exp\left[-z(p)\,\varepsilon^2\langle 4(y_0-y)\rangle^{-1}\right] d\omega$$

$$+ \psi(p)\,(4\pi)^{-\frac{N}{2}} \lim_{\varepsilon\to 0}\lim_{y\to y_0}\left\{4^{\frac{N-1}{2}}\left(\frac{N-2}{2}\right)! \int_{\omega_N} d\omega\right\}$$

$$= \frac{1}{2}\,\psi(p)\,\pi^{-\frac{N}{2}}\,\Gamma\left(\frac{N}{2}\right)\,\omega_N$$

where in the last equation it is clear that ω_N is the surface of the N-dimensional unit sphere and Γ is the well known gamma function. On the other hand for N odd we write using 2.23

2.25 $$\lim_{y\to y_0}\iint_{D(y)} \psi z \gamma_p\, ds = \psi(p)\, z(p)^{\frac{1}{2}}(4\pi)^{-\frac{N}{2}} \lim_{\varepsilon\to 0}\lim_{y\to y_0}\int_{\omega_N}\int_0^{\varepsilon} 4^{\frac{N-1}{2}}\cdot$$

$$\cdot\left\{\left(\frac{N-2}{2}\right)\left(\frac{N-4}{2}\right)\cdots\cdots\frac{1}{2}\right\}(y_0-y)^{-\frac{1}{2}} \exp\left[-z(p)\, r^2\langle 4(y_0-y)\rangle^{-1}\right] dr\, d\omega.$$

To evaluate the integral on the right of the last equation it is necessary to make the following change of variables. Let

2.26 $$\Theta = z(p)\, r^2\langle 4(y_0-y)\rangle^{-1}$$

then

2.27 $$r^2 = \frac{4}{z(p)}(y_0 - y)\,\theta$$

and

2.28 $$\lim_{y \to y_0} \int_0^{\varepsilon} (y_0 - y)^{-\frac{1}{2}} \exp\left[-z(p) r^2 \langle 4(y_0 - y)\rangle^{-1}\right] dr$$

$$= \lim_{y \to y_0} \int_0^{\frac{z(p)\varepsilon^2}{4(y_0 - y)}} z(p)^{-\frac{1}{2}} \theta^{-\frac{1}{2}} \exp(-\theta)\, d\theta$$

$$= z(p)^{-\frac{1}{2}} \int_0^{\infty} \theta^{-\frac{1}{2}} \exp(-\theta)\, d\theta$$

$$= z(p)^{-\frac{1}{2}}\, \Gamma\left(\tfrac{1}{2}\right).$$

The value of the above integral is obtained by direct calculation. Substitution of 2.28 into 2.25 now yields

2.29 $$\lim_{y \to y_0} \iint_{D(y)} \psi z \gamma_p \, ds = \tfrac{1}{2}\psi(p)(\pi)^{-\frac{N}{2}}\left(\tfrac{N-2}{2}\right)\left(\tfrac{N-4}{2}\right)\cdots\tfrac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)\omega_N$$

$$= \tfrac{1}{2}\psi(p)(\pi)^{-\frac{N}{2}}\,\Gamma\left(\tfrac{N}{2}\right)\omega_N$$

where use has been made of the well-known recurrence relation for the gamma function. The desired result is now obtained by noting from page 223 of [2]

2.30 $$\omega_N = 2\pi^{\frac{N}{2}}\left[\Gamma\left(\tfrac{N}{2}\right)\right]^{-1}$$

and substitution of this expression into 2.24 and 2.29 yields

2.31 $$\lim_{y \to y_0} \iint_{D(y)} \psi z \gamma_p \, ds = \psi(x_0, y_0)$$

for N either even or odd.

3. An Auxiliary Function for the Parabolic Problem

Here our aim is to construct a sufficiently smooth function $\mathcal{F}$ which will be employed in the following chapters to obtain solution bounds for the parabolic problem. The desired function $\mathcal{F}$ will be defined so that

2.32 $$-\mathcal{F} < \infty \qquad \text{for } (x,y) \in \overline{V}$$

and

2.33 $$\overline{J}(\mathcal{F}) \leqslant 0 \qquad \text{for } (x,y) \in \overline{V}$$

where $\overline{J}$ satisfies the conditions of the previous section. In addition to the above, it is necessary that $\mathcal{F}$ have a singularity at $p \in D(y_0)$, which is of a slightly lower order than the singularity of the parametrix $\gamma\, p$ at $p \in D(y_0)$. The desired function is constructed by first considering for $N \geqslant 1$

2.34 $$\overline{\mathcal{F}} = (y_0 - y)^{-\frac{N-1}{2}} \exp\left[-(2N-1)\, z(p) \rho^2 \left\langle 8N(y_0 - y)\right\rangle^{-1}\right]$$

and computing for $(x,y) \in V$

2.35 $$\overline{J}(\overline{\mathcal{F}}) = (y_0 - y)^{-\frac{N+3}{2}} \left[-\tfrac{1}{4N}\, a^{ij}_{,i}\, (2N-1)\, z(p)\, a_{kj}(p)\, (x^k - x_0^k)(y_0 - y)\right.$$

$$-\frac{1}{4N}(2N-1)z(p)a^{ij}a_{,ij}(p)(y_0-y)+\frac{1}{16N^2}(2N-1)^2 z(p)^2 a^{ij}a_{kj}(p)(x^k-x_0^k)a_{,il}(p)(x^l-x_0^l)$$

$$+\left(\frac{N-1}{2}\right)z(y_0-y)-\frac{1}{8N}(2N-1)z\,z(p)\rho^2+\frac{\partial z}{\partial y}(y_0-y)^2\Big]\cdot$$

$$\cdot\exp\left[-(2N-1)z(p)\rho^2\langle 8N(y_0-y)\rangle^{-1}\right]$$

$$=(y_0-y)^{-\frac{N+3}{2}}\Big[-\frac{1}{4N}a^{ij}_{,i}(2N-1)z(p)a_{kj}(p)(x^k-x_0^k)(y_0-y)-\frac{1}{4}z(p)(y_0-y)$$

$$-\frac{1}{4N}(2N-1)z(p)g_k^{ij}(x^k-x_0^k)a_{ij}(p)(y_0-y)-\frac{1}{4N}(2N-1)z(p)g_y^{ij}a_{ij}(p)(y_0-y)^2$$

$$+\left(\frac{N-1}{2}\right)h_i(x^i-x_0^i)(y_0-y)+\left(\frac{N-1}{2}\right)h_y(y_0-y)^2+\frac{\partial z}{\partial y}(y_0-y)^2$$

$$-(4N)^{-2}(2N-1)z(p)^2\rho^2+(4N)^{-2}(2N-1)^2z(p)^2g_r^{ij}(x^r-x_0^r)a_{kj}(p)(x^k-x_0^k)a_{il}(p)(x^l-x_0^l)$$

$$+(4N)^{-2}(2N-1)^2z(p)^2g_y^{ij}a_{kj}(p)(x^k-x_0^k)a_{il}(p)(x^l-x_0^l)(y_0-y)$$

$$-(8N)^{-1}(2N-1)z(p)h_i(x^i-x_0^i)\rho^2-(8N)^{-1}z(p)h_y\rho^2(y_0-y)\Big]\cdot$$

$$\cdot\exp\left[-(2N-1)z(p)\rho^2\langle 8N(y_0-y)\rangle^{-1}\right]$$

with g_k^{ij}, g_y^{ij}, h_i and h_y defined by 2.19 and 2.21. For $(x,y)\in V$ and sufficiently close to p, the second term in the brackets on the right of the last equation dominates the first and third through seventh terms while the eighth dominates the remaining terms. Due to the boundedness of the g's and h's and the piecewise continuity of $a^{ij}_{,i}$ in $\overline{V}$, it follows that there exist positive numbers δ_{σ} and ρ_0 such that

$$
2.36 \qquad \begin{cases} \bar{J}(\bar{\mathcal{F}}) \leq 0 & (x,y) \in \{x,y \mid \rho \leq \rho_0 \;,\; 0 \leq y_0 - y \leq \delta_{0\tilde{1}}\} \\ \bar{J}(\bar{\mathcal{F}}) < 0 & (x,y) \in \{x,y \mid \rho \leq \rho_0 \;,\; 0 < \varepsilon \leq y_0 - y \leq \delta_{0\tilde{1}}\}. \end{cases}
$$

Let $\delta_{0\tilde{2}}$ be such that

$$
2.37 \qquad \frac{\partial z}{\partial y}(y_0-y)^2 + \left(\frac{N-1}{2}\right)(y_0-y)\,z - \frac{(2N-1)\,z(p)\,\rho_0^2}{8N}\,z < 0
$$

for all $(x,y) \in \{x,y \mid (x,y) \in \bar{V},\ 0 \leq y_0 - y \leq \delta_{0\tilde{2}}\}$. We set

$$
2.38 \qquad \delta_0 = \min\{\delta_{0\tilde{1}}, \delta_{02}\}
$$

and then define the desired auxiliary function as follows in terms of non-negative constants α and b

$$
2.39 \qquad \mathcal{F} \begin{cases} = \bar{\mathcal{F}} + \alpha (y_0-y)^{-\frac{N-1}{2}} \exp\{-(2N-1)\,z(p)\,\rho_0^2 \langle 8N(y_0-y)\rangle^{-1}\} \\ \qquad\qquad (x,y) \in \bar{V} \;,\; 0 < y_0 - y \leq \delta_0 \\ = \delta_0^{-\frac{1}{2}(N-1+2b)} (y_0-y)^{b} \langle \exp\{-(2N-1)\,z(p)\,\rho^2 [8N(y_0-y)]^{-1}\} \\ \quad + \alpha \exp\{-(2N-1)\,z(p)\,\rho_0^2 [8N(y_0-y)]^{-1}\}\rangle \\ \qquad\qquad (x,y) \in \bar{V} \;,\; \delta_0 < y_0 - y \end{cases}
$$

The constants α and ℓ appearing in the last equation shall be explicitly chosen in the proof of the theorem which follows this paragraph. On the surface $D(y_o)-\{p\}$, the function $\mathcal{F}$ and its derivatives are defined as the limits of these functions as (x,y) approaches (x,y_o) with $(x,y) \in V$ and $x \neq x_o$; thus we have

2.40 $$\mathcal{F}(x,y_o) = \lim_{y \to y_o} \mathcal{F}(x,y) \qquad x \neq x_o$$

2.41 $$\mathcal{F}_{,i}(x,y_o) = \lim_{y \to y_o} \mathcal{F}_{,i}(x,y) \qquad x \neq x_o$$

2.42 $$\mathcal{F}_{,ij}(x,y_o) = \lim_{y \to y_o} \mathcal{F}_{,ij}(x,y) \qquad x \neq x_o$$

and

2.43 $$\frac{\partial \mathcal{F}}{\partial y}(x,y_o) = \lim_{y \to y_o} \frac{\partial \mathcal{F}}{\partial y}(x,y) \qquad x \neq x_o$$

<u>Theorem 2.2</u>. The function $\mathcal{F}$ (α and ℓ appropriately fixed) is twice piecewise continuously differentiable in x and piecewise continuously differentiable in y for $(x,y) \in \overline{V}$ except at $p = (x_o,y_o)$ and is such that

2.44 $$\mathcal{F} \geq 0 \ , \ J(\mathcal{F}) \leq 0 \qquad (x,y) \in \overline{V}$$

and

2.45 $\quad \mathcal{F} > 0\ , \ \bar{J}(\mathcal{F}) < 0 \quad (x,y) \in \{x,y \,|\, (x,y) \in \bar{V}, 0 < \varepsilon \leq y_0 - y\}$

where the coefficients of $\bar{J}$ satisfy the requirements of section 2 of this chapter and, in addition, the condition

2.46 $\quad \min_{\bar{V}} z = m_2 > 0$

Proof: The function $\mathcal{F}$ obviously satisfies the continuity and differentiability requirements of the theorem.

Our aim shall now be to establish 2.44 and 2.45 for properly chosen constants α and b. From 2.36, 2.37 and the definition of $\mathcal{F}$ it is clear that the desired inequalities are satisfied for all $\alpha \geq 0$ where $0 \leq y_0 - y \leq \delta_0$ and $\rho \leq \rho_0$. For $\rho > \rho_0$ and $0 \leq y_0 - y \leq \delta_0$ we compute by means of 2.39

2.47
$$\bar{J}(\mathcal{F}) = \bar{J}(\bar{\mathcal{F}}) + \alpha (y_0 - y)^{-\frac{N+3}{2}} \Big\{ \frac{\partial z}{\partial y} (y_0 - y)^2$$

$$+ \left(\frac{N-1}{2}\right)(y_0 - y) z - (2N-1) z(p) \rho_0^2 (8N)^{-1} z \Big\} \cdot$$

$$\cdot \exp\left\{-(2N-1) z(p) \rho_0^2 \langle 8N (y_0 - y)\rangle^{-1}\right\}.$$

For $(x,y) \in \overline{V}$ and $y_0 - y > \delta_0$ we have

2.48 $$J(\mathcal{F}) = \delta_0^{-\frac{1}{2}(N-1+2b)} \{ -(4N)^{-1} a^{ij}_{oj} (2N-1) z(p) a_{i2}(p) (x^2 - x_0^2)(y_0 - y)^{b-1}$$

$$- (4N)^{-1} a^{ij} (2N-1) z(p) a_{ij}(p) (y_0-y)^{b-1}$$

$$+ (4N)^{-2} a^{ij} (2N-1)^2 z(p)^2 a_{i2}(p)(x^2 - x_0^2) a_{oj}(p) (x^o - x_0^o)(y_0 - y)^{b-2}$$

$$- z b (y_0 - y)^{b-1} - (8N)^{-1} z (2N-1) z(p) \rho^2 (y_0 - y)^{b-2}$$

$$+ \frac{\partial z}{\partial y} (y_0 - y)^b \} \exp\{ -(2N-1) z(p) \rho^2 \langle 8N(y_0 - y) \rangle^{-1} \}$$

$$+ \alpha \{ \frac{\partial z}{\partial y} (y_0 - y)^b - z b (y_0 - y)^{b-1}$$

$$- (8N)^{-1} z (2N-1) z(p) \rho^2 (y_0 - y)^{b-2} \}.$$

$$\cdot \exp\{ -(2N-1) z(p) \rho^2 \langle 8N(y_0 - y) \rangle^{-1} \}.$$

Having calculated the above, we are now in a position to choose the non-negative numbers α and b. Since 2.37 is valid in the domain of 2.47, it is clear that $\alpha \geq 0$ may be fixed so that

2.49 $$J(\mathcal{F}) \leq 0 \qquad (x,y) \in \{ x,y \mid \rho \geq \rho_0, 0 \leq y_0 - y \leq \delta_0 \}$$

and

$$2.50 \qquad \mathcal{J}(\mathcal{F}) < o \qquad (x,y) \in \{x,y \mid \rho > \rho_o, o < \varepsilon \le y_o - y \le \delta_o\}.$$

By inspecting 2.48 it is clear that $b \geq o$ may be chosen so that

$$2.51 \qquad \mathcal{J}(\mathcal{F}) < o \qquad (x,y) \in \{x,y \mid (x,y) \in \bar{V},\ y_o - y > \delta_o\}.$$

In order to establish that the auxiliary function is non-negative in $\bar{V}$ and is strictly positive in every closed set of V, we need only consider equation 2.39.

Subject to the foregoing, it is clear that the theorem is established; however, as a final remark, we note that for certain differential operators we may choose

$$2.52 \qquad \mathcal{F} = \bar{\mathcal{F}} \qquad (x,y) \in \{x,y \mid (x,y) \in \bar{V}, o \le y_o - y \le \delta_o\}$$

since $\mathcal{J}(\bar{\mathcal{F}})$ satisfies 2.49 and 2.50. When this condition is applicable, one would of course choose $\alpha = o$ in 2.39.

The conclusions of Theorem 2.2 shall be employed in the next chapter to obtain pointwise bounds for the solution of non-linear normally parabolic problems. In the first section of Chapter IV the function $Z = Z(x,y)$ is not assumed to be strictly positive and 2.46 is replaced by

2.53 $$\min_{\overline{V}} Z \geq 0$$

and

2.54 $$Z(x,y) > 0 \qquad (x,y) \in \overline{D}(y_o)$$

Under such an assumption it is obvious that Theorem 2.2 is not valid since we would in general be unable to choose k so that 2.48 defines a strictly negative number. Because of this reason we use the fact that a^{ij} is a positive definite matrix and introduce a slight different auxiliary function. This new function is much more difficult to construct since, besides having a more complicated form, it is necessary to calculate $a_{ij}(x_o,y)$ for $o \leq y \leq y_o$. Thus, even though the revised auxiliary function could be used in the linear normally parabolic case, the function as given by 2.39 would be easier to construct since the inversion of the matrix a^{ij} is only necessary at the point $p \in D(y_o)$.

When conditions 2.53 and 2.54 replace 2.46, we define $\overline{\mathcal{F}}$ as follows:

2.55 $$\overline{\mathcal{F}} = (y_o - y)^{-\frac{N-1}{2}} \exp\left\{-(kN-1)\, z(p)\,\hat{\rho}^2 \langle 8N(y_o - y)\rangle^{-1}\right\}$$

for $(x,y) \in V$ where

2.56 $$\tilde{\rho} = a_{ij}(x_0,y)(x^i - x_0^i)(x^j - x_0^j)$$

For our present definition of $\bar{\Phi}$ we notice that 2.35 is valid with

2.57 $$\tilde{\rho} \quad \text{replacing} \quad \rho$$

and

2.58 $$a_{rs}(x_0,y) \qquad \text{replacing} \qquad a_{rs}(p).$$

In the revised form of 2.35 it is also necessary to add the following term on the right side of the equation

2.59 $$\frac{\partial \tilde{\rho}}{\partial y}^2 z\, z(p)\,(2N-1)\left\langle 8N(y_0-y)^{\frac{N+1}{2}}\right\rangle^{-1}$$

$$\cdot \exp\left\{-(2N-1)\, z(p)\,\tilde{\rho}^2 \left\langle 8N(y_0-y)\right\rangle^{-1}\right\}.$$

By exactly the same arguments as previously used, we deduce that there exist positive numbers $2\delta_{01}$ and ρ_{01} such that

2.60 $\bar{J}(\bar{\mathcal{F}}) \leq 0 \qquad (x,y) \in \{x,y \mid \tilde{\rho} \leq \rho_{01}, 0 \leq y_o - y \leq 2\delta_{01}\}$

and

2.61 $\bar{J}(\bar{\bar{\mathcal{F}}}) < 0 \qquad (x,y) \in \{x,y \mid \tilde{\rho} \leq \rho_{01}, 0 \leq y_o - y \leq 2\delta_{01}\}.$

Since $Z(x,y)$ is strictly positive on the surface $\bar{D}(y_o)$ it follows from the conditions of the first section of this chapter that there exists a positive number δ_{02} such that

2.62 $$\frac{\partial Z}{\partial y}(y_0-y)^2 + \left(\frac{N-1}{2}\right)(y_0-y)Z - \frac{(2N-1)Z(\rho)\rho_{01}}{8N} Z < 0$$

for all $(x,y) \in \{x,y \mid (x,y) \in \bar{V}, 0 \leq y_o - y \leq 2\delta_{02}\}$

We set

2.63 $$\delta_0 = \min\{\delta_{01}, \delta_{02}\}$$

and notice that

2.54 $\bar{J}(\bar{\alpha}) \leq 0 \qquad (x,y) \in \{x,y \mid (x,y) \in \bar{V}, 0 \leq y_o - y \leq 2\delta_o\}$

and

2.65 $$\bar{J}(\bar{\alpha}) < 0 \qquad (x,y) \in \{x,y \,|\, (x,y) \in \bar{V}, 0 < \varepsilon \leq y_0 - y \leq 2\delta_0\}$$

where

2.66 $$\bar{\alpha} = \alpha\,(y_0 - y)^{-\frac{N-1}{2}} \exp\{-(2N-1)\,z(p)\,\rho_{01}^2 \langle 8N(y_0-y)\rangle^{-1}\}$$

Since 2.62 is true, it is evident that 2.64 and 2.65 are valid for all $\alpha > 0$. By comparing $\bar{\Phi}$ and $\bar{\alpha}$, we observe that we may choose $\alpha = \alpha_1$, such that

2.67 $$J(\bar{\Phi} + \bar{\alpha}) \leq 0 \qquad (x,y) \in \{x,y \,|\, (x,y) \in \bar{V}, 0 \leq y_0 - y \leq 2\delta_0\}$$

and

2.68 $$\bar{J}(\bar{\Phi} + \bar{\alpha}) < 0 \qquad (x,y) \in \{x,y \,|\, (x,y) \in \bar{V}, 0 < \varepsilon \leq y_0 - y \leq 2\delta_0\}$$

Next we introduce the function $\tilde{\Phi}$ given by

2.69 $$\tilde{\Phi} = \{\delta_0^{-1} - (y_0 - y)^{-1}\} \cdot$$

$$\cdot \exp\{-(8N\delta_0)^{-1}\beta\,(2N-1)\,z(p)\,\hat{\rho}^2\}$$

$$+ \bar{\Phi} + \bar{\alpha}$$

where for the present β is an arbitrary non-negative number. To fix this arbitrary number we compute

2.70 $$J(\hat{\Phi}) = \{ \delta_0^{-1} - (y_0 - y)^{-1} \} \cdot$$

$$\cdot \{ -(4N\delta_0)^{-1} \beta (2N-1) z(p)\, a^{ij}_{ji} a_{kj} (x_0, y) (x^k - x_0^k)$$

$$-(4N\delta_0)^{-1} \beta (2N-1) z(p)\, a^{ij} a_{ij} (x_0, y)$$

$$+(4N\delta_0)^{-2} \beta^2 a^{ij} (2N-1)^2 z(p)^2 a_{ir}(x_0, y)(x^r - x_0^r) a_{sj}(x_0, y) \cdot$$

$$\cdot (x^s - x_0^s) - \frac{\partial a_{ij}}{\partial y}(x_0, y)(8N\delta_0)^{-1} \beta z (2N-1) z(p) (x^i - x_0^i)(x^j - x_0^j)$$

$$+ \frac{\partial z}{\partial y} \} \exp\{ -(8N\delta_0)^{-1} \beta (2N-1) z(p) \tilde{\rho}^2 \}$$

$$-(y_0 - y) z \exp\{ -(8N\delta_0)^{-1} \beta (2N-1) z(p) \tilde{\rho}^2 \}$$

$$+ \bar{J}(\bar{\Phi} + \bar{\alpha}).$$

For $\delta_0 \leq y_0 - y \leq 2\delta_0$ and $(x,y) \in \overline{V}$, the last term on the right of the above equation is a strictly negative number. Next we consider the following limit

2.71
$$\lim_{\substack{\tilde{\rho}\to 0\\ 0\le y\le y_0-\delta_0}} \bar{J}(\tilde{\Phi}) = \{\delta_0^{-1}-(y_0-y)^{-1}\}\cdot$$

$$\cdot\{-(4\delta_0)^{-1}\beta(2N-1)\,z(p) + \frac{\partial z}{\partial y}\}$$

$$-(y_0-y)^{-2}z + \lim_{\substack{\tilde{\rho}\to 0\\ 0\le y\le y_0-\delta_0}} \bar{J}(\Phi+\bar{\alpha}).$$

By the remark following 2.70 and due to the form of 2.71, we deduce that there exist positive numbers ρ_{02} and β_1 such that for

2.72
$$\beta = \beta_1$$

we have

2.73
$$\bar{J}(\tilde{\Phi}) < 0 \qquad (x,y)\in\{x,y \mid \tilde{\rho}\le\rho_{02},\ 0<y<y_0-\delta_0\}.$$

We are now in a position to define the desired auxiliary function. As was mentioned above, this function will be used when bounds for the solution of a degenerate problem are sought. The function is explicitly given by

$$
2.74 \qquad \mathcal{F}
\begin{cases}
= \bar{\mathcal{F}} + \bar{\alpha} & (x,y) \in V,\ 0 < y_0 - y \le \delta_0 \\
= \tilde{\mathcal{F}} & \tilde{\rho} \le \rho_{02},\ 0 \le y \le y_0 - \delta_0 \\
= \{ \delta_0^{-1} - (y_0 - y)^{-1} \} \{ -(4N\delta_0 d)^{-1}(2N-1) z(p) \beta_1 \left(\frac{\tilde{\rho}}{\rho_{02}}\right)^d \rho_{02}^2 & \\
\quad + (4N\delta_0 d)^{-1}(2N-1) z(p) \beta_1 \rho_{02}^2 + 1 \} \cdot & \\
\quad \cdot \exp\{-(8N\delta_0)^{-1} \beta_1 (2N-1)\, z(p) \rho_{02}^2\} & \\
\quad + \bar{\mathcal{F}} + \bar{\alpha} & \tilde{\rho} > \rho_{02},\ 0 \le y \le y_0 - \delta_0
\end{cases}
$$

On the surface $D(y_0) - \{p\}$, the function $\mathcal{F}$ and its derivatives are defined as the limits of these functions as (x,y) approaches (x,y_0) with $(x,y) \in V$ and $x \neq x_0$. We now show that with d properly chosen the function $\mathcal{F}$ satisfies the conclusions of the following theorem.

<u>Theorem</u> <u>2.3</u>. The function $\mathcal{F}$, defined by 2.74, is twice piecewise continuously differentiable in x and piecewise continuously differentiable in y for $(x,y) \in \bar{V}$ except at $p = (x_0, y_0)$ and is such that

2.75 $\quad \bar{J}(\mathcal{F}) \leq o \qquad (x,y) \in \bar{V}$

2.76 $\quad \bar{J}(\mathcal{F}) < o \qquad (x,y) \in \left\{x,y \,\middle|\, (x,y) \in \bar{V}, o < \varepsilon \leq y_o - y\right\}$

and

2.77 $$\max_{\bar{V}} \left\{-\mathcal{F}\right\} = M$$

where M is a finite number. The coefficients of the differential operator $\bar{J}$ satisfy the requirements of section 2 and in addition the condition that the partial derivative of $a_{rs}(x_o, y)$ with respect to y is piecewise continuous on the line $(x_o, y) \in \bar{V}$. The function $\mathcal{Z}$ is such that

2.78 $$\min_{\bar{V}} \mathcal{Z} \geqslant o$$

and

2.79 $\quad \mathcal{Z}(x,y) > o \qquad (x,y) \in \bar{D}(y_o)$.

Proof: The continuity and differentiability of $\mathcal{F}$ are obvious for $(x,y) \in \left\{ x,y \,\middle|\, (x,y) \in \bar{V},\ o \leq y_o - y \leq \delta_o \right\}$. For the lower part of $\bar{V}$ we calculate

2.80
$$\{\mathcal{F}\}_{\substack{\tilde{\rho}=\rho_{02} \\ 0\le y\le y_0-\delta_0}} = \{\delta_0^{-1}-(y_0-y)^{-1}\}\cdot$$

$$\cdot \exp\{-(8N\delta_0)^{-1}\beta_1(2N-1)z(p)\rho_{02}^2\} + \overline{\mathcal{F}} + \bar{\alpha}$$

$$= \lim_{\substack{\tilde{\rho}\to\rho_{02} \\ \tilde{\rho}>\rho_{02} \\ 0\le y\le y_0-\delta_0}} \{\mathcal{F}\}$$

and

2.81
$$\{\mathcal{F}_{,i}\}_{\substack{\tilde{\rho}=\rho_{02} \\ 0\le y\le y_0-\delta_0}} = \{\delta_0^{-1}-(y_0-y)^{-1}\}\cdot$$

$$\cdot\{-(4N\delta_0)^{-1}\beta_1(2N-1)z(p)\,a_{ik}(x_0,y)(x^k-x_0^k)\}\cdot$$

$$\cdot \exp\{-(8N\delta_0)^{-1}\beta_1(2N-1)z(p)\rho_{02}^2\} + \overline{\mathcal{F}}_{,i}$$

$$= \lim_{\substack{\tilde{\rho}\to\rho_{02} \\ \tilde{\rho}>\rho_{02} \\ 0\le y\le y_0-\delta_0}} \{\mathcal{F}_{,i}\}.$$

Our task is next to select the number α so that $\mathcal{F}$ satisfies 2.75 and 2.76. From the remarks associated with equations 2.55 to 2.73 it is clear that these inequalities are satisfied on the set

2.82 $$\{x,y \mid (x,y)\in \bar{V},\ 0\le y_0-y\le \delta_0\}\cup\{x,y \mid \tilde{\rho}\le \rho_{02},\ 0\le y\le y_0-\delta_0\}$$

For $\rho > \rho_{02}$ and $0 \leq y \leq y_0 - \delta_0$ we compute

2.83 $$\bar{J}(\bar{\Phi})=\{\delta_0^{-1}-(y_0-y)^{-1}\}\{(4N\delta_0)^{-1}(2N-1)z(\rho)\beta_1\}\{-a^{ij}a_{ij}(x_0,y)\left(\frac{\tilde{\rho}}{\rho_{02}}\right)^{d-2}$$

$$-a^{ij}_{y}a_{is}(x_0,y)(x^s-x_0^s)\left(\frac{\tilde{\rho}}{\rho_{02}}\right)^{d-2}-z\frac{\partial a_{n0}(x_0,y)}{\partial y}(x^2-x_0^2)(x^s-x_0^s)\left(\frac{\tilde{\rho}}{\rho_{02}}\right)^{d-1}\rho_{02}$$

$$-(d-2)\,a^{ij}a_{is}(x_0,y)(x^s-x_0^s)a_{nj}(x_0,y)(x^2-x_0^2)\left(\frac{\tilde{\rho}}{\rho_{02}}\right)^{d-4}\rho_{02}^{-2}\}$$

$$\cdot\exp\{-(8N\delta_0)^{-1}\beta_1(2N-1)z(\rho)\beta_2^2\}+\{-(y_0-y)^{-2}z+$$

$$+[\delta_0^{-1}-(y_0-y)^{-1}]\frac{\partial z}{\partial y}\}\{-(4N\delta_0 d)^{-1}(2N-1)z(\rho)\beta_1\left(\frac{\tilde{\rho}}{\rho_{02}}\right)^{d}\rho_{02}^2$$

$$+(4N\delta_0 d)^{-1}(2N-1)z(\rho)\beta_1\rho_{02}^2+1\}\cdot$$

$$\cdot\exp\{-(8N\delta_0)^{-1}\beta_1(2N-1)z(\rho)\rho_{02}^2\}+\bar{J}(\bar{\Phi}+\bar{\alpha}).$$

In order to determine an appropriate choice of the constant d we note that

2.84 $$a^{ij}a_{ir}(x_0,y)(x^r-x_0^r)a_{sj}(x_0,y)(x^s-x_0^s)\geq a_0 a_1^{-1}\rho_{02}^2$$

for all (x,y) belonging to the domain of definition of 2.83; hence it is evident that d may be chosen so that 2.83 defines a strictly negative number.

With d so chosen it is obvious that the finite

number M given by

2.85 $$M = \max_{\overline{V}} \{ -\mathcal{F} \}$$

exists since $\mathcal{F}$ is bounded everywhere on $\overline{V}$ except at the point p and at this point one has

2.86 $$\lim_{\substack{\bar{p} \to p \\ \bar{p} \in \overline{V}}} \mathcal{F} \geq 0.$$

In future investigations bearing on the subject matter of this thesis, an attempt will be made to replace 2.79 by

2.87 $$\Xi(p) > 0.$$

Next we shall establish the existence of an integral which will often appear in the bounds which are developed in the following chapters.

Theorem 2.4. The function $\mathcal{F}$ is such that the integral

2.88 $$-\iiint_V [J(\mathcal{F})]^{-1} [J(\gamma_p)]^2 \, dV$$

exists.

Proof: As in the other theorems of this section, the proof is completed by discussing two cases. When 2.46 is satisfied, it follows from 2.35 that there exist positive numbers δ_1 and ρ_1 which are such that

2.89 $$-\bar{J}(\mathcal{F}) \geqslant (y_o-y)^{-\frac{N+3}{2}}\left[\frac{1}{8}z(p)(y_o-y)+(32N^2)^{-1}(2N-1)z(p)^2\rho^2\right]\cdot$$
$$\cdot\exp\left\{\left[-8N(y_o-y)\right]^{-1}\left[(2N-1)z(p)\rho^2\right]\right\}$$

for $\rho \leqslant \rho_1 \leqslant \rho_o$ and $o \leqslant y_o-y \leqslant \delta_1 \leqslant \delta_o$. With ρ^2 replaced by $\tilde{\rho}^2$ it is evident that a similar result follows for $\mathcal{F}$ in case 2.79 is applicable. Hence we have

2.90 $$-\bar{J}(\mathcal{F}) \geqslant \frac{1}{8}(y_o-y)^{-\frac{N+1}{2}}z(p)\exp\left\{-\left[8N(y_o-y)\right]^{-1}\left[(2N-1)z(p)\rho^2\right]\right\}$$

for $\rho, \tilde{\rho} \leqslant \rho_1 \leqslant \rho_o$ and $o \leqslant y_o-y \leqslant \delta_1 \leqslant \delta_o$. Next we consider

2.91 $$\iiint_V \left[-\bar{J}(\mathcal{F})\right]^{-1}\left[\bar{J}(\gamma_p)\right]^2 dV$$
$$= \iiint_{V-V_1}\left[-\bar{J}(\mathcal{F})\right]^{-1}\left[\bar{J}(\gamma_p)\right]^2 dV + \iiint_{V_1}\left[-\bar{J}(\mathcal{F})\right]^{-1}\left[\bar{J}(\gamma_p)\right]^2 dV$$

where

2.92 $$V_1 = \left\{(x,y) \;\middle|\; \rho, \tilde{\rho} \leqslant \rho_1,\ o \leqslant y_o-y \leqslant \delta_1\right\}.$$

The first integral on the right of 2.91 clearly exists since its integrand is bounded over the domain of integration; also, since

2.93 $$(8N)^{-1}(2N-1) < \frac{1}{2},$$

the second integral on the right of the aforesaid equation is seen to be finite by application of 2.90 and Theorem 2.1.

3. Generalized Form of an Identity of Hörmander

The identity under consideration was derived by Hörmander [16] for the purpose of obtaining a uniqueness theorem and estimates for normally hyperbolic partial differential equations. Essentially the same identity was employed by Payne and Weinberger [21] to obtain solution bounds for second order elliptic equations. In what follows a slightly generalized form of the same identity is derived.

Let $f^1(x,y), \ldots f^N(x,y)$ be any set of functions which are piecewise continuously differentiable in x for $(x,y) \in \overline{V}$. By direct differentiation and application of the divergence theorem, we have for any function ψ which is continuously differentiable in x for $(x,y) \in V$ and whose second derivative in x is continuous in the interior of a finite number of subregions, the sum of which is D(y), where y is such that $o < y < y_o$ (in the above we suppose that D(y) is a domain of N dimensional space).

2.94
$$\iint_S \left[f^k a^{ij} - f^i a^{kj} - f^j a^{ik}\right] \psi_{,i}\, \psi_{,j}\, n_k \, ds$$

$$= -2 \iiint_V f^k \psi_{,k} \left(a^{ij} \psi_{,i}\right)_{,j} \, dV$$

$$+ \iiint_V \left[f^k_{,k} a^{ij} - f^i_{,k} a^{kj} - f^j_{,k} a^{ij} + f^k a^{ij}_{,k}\right] \psi_{,i}\, \psi_{,j}\, dV.$$

The functions $a^{ij} = a^{ij}(x,y)$, $i,j = 1, 2 \ldots N$, which appear in the last equation, are assumed to be symmetric, piecewise continuously differentiable for $(x,y) \in \overline{V}$ and such that 2.7 is satisfied. (For the results of this section the piecewise continuity of the derivative of a^{ij} with respect to y is not needed; but, since this condition shall later be necessary, it is included here.)

Let

2.95 $$\frac{\partial\psi}{\partial\nu} = a^{ij}\psi_{,i}n_j \qquad (x,y) \in S$$

and then note that the vector

2.96 $$(\tau^j, 0) = \left(a^{ij}\left[\psi_{,i} - \frac{\partial\psi}{\partial\nu}n_i\{a^{pq}n_pn_q\}^{-1}\right], 0\right)$$

defined almost everywhere on S is orthogonal to (n_i, n_y) where in 2.96 we have made use of 2.5 and 2.7. From the above it is clear that (τ^j, o) is a tangent vector to the surface S, is perpendicular to the vector $(0,1)$ and is defined almost everywhere on S. Following closely the procedure of section 2 of [21] we renormalize by setting

2.97 $$(t^i, o) = (\tau^i[a^{pq}n_pn_q]^{\frac{1}{2}}[a_{rs}\tau^r\tau^s]^{-\frac{1}{2}}, o)$$

where a_{ij} is the inverse matrix of a^{ij}. For $(x,y) \in S$ we have

2.98 $$a_{ij}t^it^j = a^{ij}n_in_j = n$$

where n is defined by this equation. Due to the above, the directional derivative on S,

2.99 $$\frac{\partial\psi}{\partial t} = \psi_{,i}\, t^i + \frac{\partial\psi}{\partial y}\cdot 0 = \psi_{,i}\, t^i \quad ,$$

is in a tangential direction which is perpendicular to the "y" direction. For $(x,y)\in S$ we notice that

2.100 $$a_{ij}\tau^i\tau^j = a_{ij}a^{ki}a^{lj}\left[\psi_{,k} - \frac{\partial\psi}{\partial\nu}\frac{n_k}{n}\right]\left[\psi_{,l} - \frac{\partial\psi}{\partial\nu}\frac{n_l}{n}\right]$$

$$= \tau^k\left[\psi_{,k} - \frac{\partial\psi}{\partial\nu}\frac{n_k}{n}\right]$$

$$= \tau^k\,\psi_{,k} \quad .$$

Also

2.101 $$a_{ij}\tau^i\tau^j = a^{ij}\left[\psi_{,j}\psi_{,i} - \frac{\partial\psi}{\partial\nu}\left(\frac{\psi_{,i}n_j + \psi_{,j}n_i}{n}\right) + \left(\frac{\partial\psi}{\partial\nu}\right)^2\frac{n_j n_i}{n^2}\right]$$

$$= a^{ij}\psi_{,j}\psi_{,i} - \frac{1}{n}\left(\frac{\partial\psi}{\partial\nu}\right)^2 \quad .$$

Hence almost everywhere for $(x,y)\in S$

2.102 $$a^{ij}\psi_{,i}\psi_{,j} = n^{-1}\left[\left(\frac{\partial\psi}{\partial\nu}\right)^2 + \frac{\partial\psi}{\partial t}\left(\psi_{,j}\tau^j\right)^{\frac{1}{2}} n^{\frac{1}{2}}\right] = n^{-1}\left[\left(\frac{\partial\psi}{\partial\nu}\right)^2 + \left(\frac{\partial\psi}{\partial t}\right)^{3/2}\cdot\right.$$

$$\left.\cdot\left(a_{rs}\tau^r\tau^s\right)^{1/4} n^{1/4}\right] = n^{-1}\left[\left(\frac{\partial\psi}{\partial\nu}\right)^2 + \left(\frac{\partial\psi}{\partial t}\right)^{7/4}\left(a_{rs}\tau^r\tau^s\right)^{1/8} n^{1/8}\right] = n^{-1}\left[\left(\frac{\partial\psi}{\partial\nu}\right)^2 + \left(\frac{\partial\psi}{\partial t}\right)^2\right] .$$

Moreover almost everywhere on S we have

2.103 $$a_{ij} f^i t^j \frac{\partial \psi}{\partial t} = a_{ij} f^i T^j \psi_{,k} T^k \left[\frac{n}{\psi_{,r} T^r} \right]$$

$$= a_{ij} f^i a^{kj} \left[\psi_{,k} - \frac{\partial \psi}{\partial \nu} \frac{n_k}{m} \right] n$$

$$= n f^i \psi_{,i} - f^i n_i \frac{\partial \psi}{\partial \nu}$$

and thus

2.104 $$f^i n_i \frac{\partial \psi}{\partial \nu} + a_{ij} f^i t^j \frac{\partial \psi}{\partial t} = n f^i \psi_{,i} .$$

Using 2.102 and 2.104 we rewrite 2.94 as

2.105 $$\iint_S n^{-1} \left[f^k n_k \left\{ \left(\frac{\partial \psi}{\partial t} \right)^2 - \left(\frac{\partial \psi}{\partial \nu} \right)^2 \right\} - 2 a_{ij} f^i t^j \frac{\partial \psi}{\partial \nu} \frac{\partial \psi}{\partial t} \right] dS$$

$$= -2 \iiint_V f^k \psi_{,k} \left(a^{ij} \psi_{,i} \right)_{,j} dV$$

$$+ \iiint_V \left[f^k_{,k} a^{ij} - f^i_{,k} a^{kj} - f^j_{,k} a^{ik} + f^k a^{ij}_{,k} \right] \psi_{,i} \psi_{,j} \, dV.$$

We now choose f^k, $k = 1, 2 \ldots N$, so that a bound for the square of the integral of the conormal type derivative 2.95 may be obtained from 2.105. Previously we required that these auxiliary functions be piecewise continuously differentiable in x; we now further assume that $f^i n_i$, is bounded and has a positive minimum on S. Such a set of functions was considered by Payne and Weinberger in section 3 of [21]. From this reference we note that, if S is starshaped in x with respect to some line perpendicular to the x plane, then we can choose this line as the x origin and take $f^k = x^k$.

Also if V is a domain between two boundaries each of which is starshaped in x with respect to the above perpendicular line and if V contains a cylindrical shell of radius r_o from this line then we may take $f^k = x^k\,(r-r_o)$. For further information bearing on this choice of the auxiliary functions, the reader is referred to Appendix A of [17].* With f^i as defined above, set

2.106 $$m_3 = \min_S n^{-1} f^k m_k \; .$$

Since a^{ij} is positive definite, there exists a constant c_1 which satisfies

2.107 $$-\left[f,{}_k^k a^{ij} - f,{}_k^i a^{jk} - f,{}_k^j a^{ik} + f^k a,{}_k^{ij}\right]\psi_{,i}\psi_{,j} \le C_1 a^{ij}\psi_{,i}\psi_{,j}$$

for all $(x,y) \in V$. (C_1 is an upper bound for the largest eigenvalue $\tilde{C}_1$ of the coefficient matrix on the left with respect to a^{ij}.) For a crude upper bound for $\tilde{C}_1$ we define

2.108 $$\ell^{ij} = -\left[f,{}_k^k a^{ij} - f,{}_k^i a^{jk} - f,{}_k^j a^{ik} + f^k a,{}_k^{ij}\right]$$

then

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*See also, Bramble, J. and Hubbard, B., Some Higher Order Integral Identities with Application to Bounding Techniques, in print.

2.109 $$\tilde{C}_1 = \max_{V,\xi} \frac{b^{ij}\xi_i\xi_j}{a^{ij}\xi_i\xi_j} = \frac{\bar{b}^{ij}\bar{\xi}_i\bar{\xi}_j}{\bar{a}^{ij}\bar{\xi}_i\bar{\xi}_j}$$

$$\leqslant \frac{\left\{\sum_j\left\langle\left[\sum_i(\bar{b}^{ij})^2\right]\left[\sum_i \bar{\xi}_i^2\right]\right\rangle\right\}^{1/2}\left\{\sum_j \bar{\xi}_j^2\right\}^{1/2}}{a_0\sum_i \bar{\xi}_i^2}$$

$$\leqslant \frac{B}{a_0} = C_1$$

where

2.110 $$B = \max_V\left\{\sum_{ij}(b^{ij})^2\right\}.$$

Substitution of 2.106 and 2.107 into 2.105 yields after applying the indicated inequality to the resulting expression

2.111 $$(1-\alpha)m_3\iint_S\left(\frac{\partial\psi}{\partial\nu}\right)^2 ds \leqslant \frac{1}{\alpha}\iint_S\left(nf^k m_k\right)^{-1}\left(a_{ij}f^i t^j\frac{\partial\psi}{\partial t}\right)^2 ds$$

$$+\iint_S n^{-1}f^k m_k\left(\frac{\partial\psi}{\partial t}\right)^2 ds + 2\iiint_V f^k\psi_{,k}\left(a^{ij}\psi_{,i}\right)_{,j}\,dV$$

$$+ C_1\iiint_V a^{ij}\psi_{,i}\psi_{,j}\,dV$$

where α is an arbitrary positive number. By setting

2.112 $$\alpha = \frac{1}{2}\,,$$

we may write

2.113 $$\iint_S \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds \leq \mathcal{H} + \frac{4}{m_3} \iiint_V f^k \psi_{,k} (a^{ij}\psi_{,i})_{,j}\, dv + \frac{2C_1}{m_3} \iiint_V a^{ij}\, \psi_{,i}\, \psi_{,j}\, dv$$

where

2.114 $$\mathcal{H} = \frac{4}{m_3} \iint_S (n f^k n_k)^{-1} \left(a_{ij}\, f^i t^j \frac{\partial \psi}{\partial t}\right)^2 ds + \frac{2}{m_3} \iint n^{-1} f^k n_k \left(\frac{\partial \psi}{\partial t}\right)^2 ds$$

and clearly

2.115 $$\mathcal{H} = 0$$

if $\psi = 0$ for $(x,y) \in S$. In the following chapters great use shall be made of 2.113 when the boundary value problem under consideration is of the Dirichlet type.

CHAPTER III

NON-LINEAR NORMALLY PARABOLIC PROBLEMS

1. The Non-Linear Mixed Problem

In this section, bounds for the solution function of a boundary value problem, which is normally parabolic, will be derived where the solution function is initially given and thereafter satisfies a non-linear Neumann condition on the boundary and a given differential equation in the interior.

We assume that V is an N + 1 dimensional domain with boundary $D(y_o) \cup \tilde{S} \cup D(o)$ which is such that the conditions of the first section of Chapter II are satisfied. Our goal is to determine pointwise bounds for the solution of

3.1
$$\begin{cases} J(w) = (a^{ij} w_{,i})_{,j} - z \dfrac{\partial w}{\partial y} = f(x, y, w, \nabla w) & (x,y) \in V \\ & -\infty < w, \nabla w < \infty \\ w(x,o) = g(x) & (x,y) \in D(o) \\ \dfrac{\partial w}{\partial \nu} = a^{ij} w_{,i} n_j = \ell(x, y, w) & (x,y) \in S \\ & -\infty < w < \infty \end{cases}$$

where w is continuous in $V \cup D(y_o)$, continuously differentiable in x for $(x,y) \in V$ and is such that its second derivative in x is continuous in the interior of a finite number of subregions, the sum of which is $D(y)$, where y satisfies

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$o < y < y_o$. Also we suppose that the first derivative of w in y is continuous in the interior of a finite number of subintervals, the sum of which is the line $(\bar{x}, y)$ for every fixed $\bar{x}$ where $(\bar{x}, y) \in V$. In addition it is necessary that the solution of 3.1 satisfy

$$
3.2 \qquad \begin{cases} \lim\limits_{\substack{(x,y)\to(x,o)\\(x,y)\in V}} w(x,y) = g(x) \\[2ex] \lim\limits_{\substack{(x,y)\to(\bar{x},y)\\(x,y)\in V\\(\bar{x},y)\in S}} a^{ij}(x,y)w,_i(x,y) = a^{ij}(\bar{x},y)w,_i(\bar{x},y) \qquad j = 1, 2, \ldots N. \end{cases}
$$

The boundary value problem is assumed to be such that the components of the symmetric matrix $a^{ij} = a^{ij}(x,y)$ are piecewise continuously differentiable in x and are such that 2.7 is satisfied in $\bar{V}$. In the closed set $\bar{V}$ the function $Z = Z(x,y)$ is piecewise continuously differentiable in y and satisfies

$$
3.3 \qquad \min_{\bar{V}} Z(x,y) = m_2 > o.
$$

The boundary data f, ℓ and g are assumed to be integrable and square integrable over their domains of definition for any bounded continuous function w which is continuously differentiable in x for $(x,y) \in \bar{V}$. In addition f and ℓ are required to satisfy a Lipschitz condition in all but their first two arguments. Hence there exists positive numbers $\tilde{M}_1$, $\tilde{M}_2$, $\tilde{M}^i$, $i = 1, 2 \ldots N$ such that

$$
3.4 \quad \begin{cases} |f(x,y,w_1,\nabla w_1)-f(x,y,w_2,\nabla w_2)| \leq \tilde{M}_1 |w_1-w_2| + \tilde{M}^i |w_{1,i}-w_{2,i}| \\ \qquad\qquad (x,y)\in \bar{V}\,;\ -\infty < w, \nabla w < \infty \\ |\ell(x,y,w_1)-\ell(x,y,w_2)| \leq \tilde{M}_2 |w_1-w_2| \\ \qquad\qquad (x,y)\in \bar{S}\,;\ -\infty < w < \infty . \end{cases}
$$

Except for the fact that $\ell = \ell(x,y,w)$ is required to satisfy a Lipschitz condition in w, the problem of this section is a generalization of a problem which Friedman considered in [11]. In that paper the elliptic part of the differential equation was the usual Laplace operator, f was identically zero and ℓ was a monotone non-increasing function of w. Under these restrictions the solution of the boundary value problem was shown to exist by means of the Schauder Fixed Point Theorem [26]. In a concluding remark the author indicated that in a later paper he would attempt to extend the existence proof to general second order parabolic equations. Whether or not such an extension has been carried out is not known. The existence of a solution for a similar problem was however established by Piskorek [22] and also by Pogorzelski [23].

Let $p \in D(y_0)$ be the point at which a bound for the solution is desired. We assume that the functions a^{ij} and z satisfy a Lipschitz condition at $p \in D(y_0)$ for $(x,y) \in \bar{V}$,

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that is, we assume that 2.8 and 2.9 are satisfied. In addition it is necessary to suppose that $f(x,y,w,\nabla w)$ is bounded in some neighborhood of the point p which is contained in $\overline{V}$ for any continuous function w which is continuously differentiable in x for $(x,y) \in \overline{V}$. Hence for any such function w there exists numbers δ_2, ρ_2, and $\tilde{\mathcal{H}}(w)$ all greater than zero such that

3.5 $$f(x,y,w,\nabla w) \leq \tilde{\mathcal{H}}(w)$$

for $\rho \leq \rho_2$ and $y_o - \delta_2 \leq y \leq y_o$.

The desired bound for the solution is obtained by choosing an arbitrary function $\varphi(x,y)$ which is such that $J(\varphi)$, $\frac{\partial \varphi}{\partial \nu}$ and $\varphi(x,o)$ approximate $f(x,y,\varphi,\nabla\varphi)$, $\ell(x,y,\varphi)$ and $g(x)$ respectively where we further require that φ be twice piecewise continuously differentiable in x and piecewise continuously differentiable in y for $(x,y) \in \overline{V}$.

Let

3.6 $$\psi(x,y) = w(x,y) - \varphi(x,y)$$

and compute

3.7 $$\begin{cases} J(\psi) = f(x,y,w,\nabla w) - f(x,y,\varphi,\nabla\varphi) + F(x,y) & (x,y) \in V \;; -\infty < w, \varphi, \nabla w, \nabla\varphi < \infty \\ \psi(x,o) = G(x) & (x,y) \in D(o) \\ \frac{\partial \psi}{\partial \nu} = \ell(x,y,w) - \ell(x,y,\varphi) + L(x,y) & (x,y) \in S \;; -\infty < w, \varphi < \infty \end{cases}$$

where

3.8 $$F(x,y) = f(x,y,\varphi,\nabla\varphi) - J(\varphi)$$

and

3.9 $$L(x,y) = \ell(x,y,\varphi) - \frac{\partial\varphi}{\partial\nu}\,.$$

Due to 3.3 and 3.4 we may write

3.10 $$J(\psi) \leqslant \tilde{M}_1|\psi| + \tilde{M}^i|\psi_{,i}| + F(x,y)$$

and

3.11 $$\frac{\partial\psi}{\partial\nu} \leqslant \tilde{M}_2|\psi| + L(x,y).$$

With $\bar{J}(\psi)$ and γ_p defined by 2.6 and 2.10 respectively, from the divergence theorem it follows that

3.12 $$\iiint\limits_{V(y)} \{\gamma_p J(\psi) - \psi\bar{J}(\gamma_p)\}\,dv$$

$$= \iint\limits_{S(y)} \{\gamma_p \frac{\partial\psi}{\partial\nu} - \psi\frac{\partial\gamma_p}{\partial\nu}\}\,ds - \iint\limits_{D(y)} \gamma_p Z\psi\,ds$$

$$- \iint\limits_{S(y)} n_y\gamma_p Z\psi\,ds + \iint\limits_{D(0)} \gamma_p Z\psi\,ds$$

where

3.13 $$V(y) = \left\{(x,\eta)\,\middle|\,(x,\eta)\in D(\eta),\ 0<\eta<y\right\}$$

and

3.14 $$S(y) = \left\{(x,\eta)\,\middle|\,(x,\eta)\in \bar{D}(\eta)-D(\eta),\ 0<\eta<y\right\}.$$

In 3.14 we let y approach y_0 and use Theorem 2.1 to obtain

3.15 $$\psi(x_0,y_0) = \lim_{y\to y_0} \iiint_{V(y)} \{\psi \bar{J}(\gamma_p) - \gamma_p J(\psi)\}\, dV$$

$$+ \iint_S \{\gamma_p \frac{\partial \psi}{\partial \nu} - \psi \frac{\partial \gamma_p}{\partial \nu}\} ds - \iint_S n_y \gamma_p z \psi\, ds + \iint_{D(0)} \gamma_p z \psi\, ds$$

which becomes after substitution of equations 3.7 to 3.11

3.16 $$|w(x_0,y) - \chi(p)|$$

$$= \Big| \lim_{y\to y_0} \iiint_{V(y)} \psi \bar{J}(\gamma_p) dV - \lim_{y\to y_0} \iiint_{V(y)} \gamma_p \{f(x,y,w,\nabla w) - f(x,y,\varphi,\nabla\varphi)\} dV$$

$$+ \iint_S \gamma_p \{\ell(x,y,w) - \ell(x,y,\varphi)\} ds$$

$$- \iint_S \psi \frac{\partial \gamma_p}{\partial \nu} ds - \iint_S n_y \gamma_p z \psi\, ds \Big|$$

$$\leq \Big| \lim_{y\to y_0} \iiint_{V(y)} \psi \bar{J}(\gamma_p) dV \Big| + \Big| \lim_{y\to y_0} \iiint_{V(y)} \gamma_p \{f(x,y,w,\nabla w) - f(x,y,\varphi,\nabla\varphi)\} dV \Big|$$

$$+ \Big\{ \iint_S \psi^2 ds \Big\}^{1/2} \Big\{ \Big\langle \iint_S \Big(\frac{\partial \gamma_p}{\partial \nu} + n_y z \gamma_p \Big)^2 ds \Big\rangle^{1/2} + \tilde{M}_2 \Big\langle \iint_S \gamma_p^2 ds \Big\rangle^{1/2} \Big\}$$

where

3.17 $$\chi(p) = \varphi(x_0,y_0) + \iint_S L(x,y) \gamma_p\, ds$$

$$+ \iint_{D(0)} \gamma_p z G\, ds - \lim_{y\to y_0} \iiint_{V(y)} \gamma_p F\, dV.$$

The terms on the right of 3.17 are all known; however, it is necessary to prove that the last integral converges. For this purpose we write

3.18
$$\lim_{y \to y_0} \iiint_{V(y)} \gamma_p F \, dV$$

$$= \lim_{y \to y_0} \left\{ \iiint_{V(y)-V(\rho_2,\delta_2)} \gamma_p F \, dv + \iiint_{V(y)\cap V(\rho_2,\delta_2)} \gamma_p F \, dv \right\}$$

$$= \iiint_{V-V(\rho_2,\delta_2)} \gamma_p f(x,y,\varphi,\nabla\varphi) \, dv - \iiint_{V-V(\rho_2,\delta_2)} \gamma_p J(\varphi) \, dv$$

$$+ \lim_{y \to y_0} \iiint_{V(y)\cap V(\rho_2,\delta_2)} \gamma_p \{ f(x,y,\varphi,\nabla\varphi) - J(\varphi) \} \, dv$$

where

3.19
$$V(\rho_2, \delta_2) = \{ (x,y) \mid \rho \le \rho_2 \ , \ 0 \le y_0 - y \le \delta_2 \}$$

and on the surface $D(y_o)-\{p\}$ it is clear that γ_p is defined as indicated in Section 2 of Chapter II. We now consider the three integrals on the extreme right of 3.18. The first exists since γ_p is bounded in the domain of integration and f is integrable over V. The second integral exists since its integrand is bounded over the domain of integration. For the last term we use the fact that $[f(x,y,\varphi,\nabla\varphi)-J(\varphi)]$ is bounded over $V(\rho_2, \delta_2)$ due to 3.5 and the definition of J; thus the integral exists since γ_p is easily seen to be positive and integrable over $V(\rho_2, \delta_2)$.

From the above remarks it is clear that all the terms on the left of 3.16 except $w(x_o,y_o)$ are known and hence to

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obtain a bound for the solution it is only necessary to estimate the expressions on the right. To obtain this result, we introduce the function $\tilde{\Psi}$ defined by

3.20 $$\Psi = \tilde{\Psi} \exp\left[-K(y_0-y)\right]$$

and calculate

3.21 $$J(\Psi) = J(\tilde{\Psi} \exp\left[-K(y_0-y)\right]) = \tilde{J}(\tilde{\Psi}) \exp\left[-K(y_0-y)\right]$$

where

3.22 $$\tilde{J}(\tilde{\Psi}) = J(\tilde{\Psi}) - Kz\tilde{\Psi}$$

Let f^i, $i = 1, 2 \ldots N$, be the set of auxiliary functions which was introduced in Section 4 of Chapter II, that is, the f^i are piecewise continuously differentiable in x for $(x,y) \in \bar{V}$ and are such that

3.23 $$0 < m_4 = \min_s f^k n_k.$$

With the f^i as thus defined we may write

3.24 $$\iint_S \tilde{\Psi}^2 ds \leq \frac{1}{m_4} \iint_S \tilde{\Psi}^2 f^i n_i \, ds$$

$$= \frac{2}{m_4} \iiint_V \tilde{\Psi} \tilde{\Psi}_{,i} f^i dV + \frac{1}{m_4} \iiint_V \tilde{\Psi}^2 f^i_{,i} \, dV$$

$$\leq \frac{\alpha_1 M_1}{m_4 a_0} \iiint_V a^{ij} \tilde{\Psi}_{,i} \tilde{\Psi}_{,j} \, dV + \left(\frac{1}{\alpha_1 m_4} + \frac{M_2}{m_4}\right) \iiint_V \tilde{\Psi}^2 dV$$

where α_1 is any positive number and

3.25 $$M_1 = \max\left\{\sum_1^N f^i f^i\right\}$$

3.26 $$M_2 = \max_V \left\{ f^{\,i}_{,i} \right\}.$$

In arriving at 3.24 we have made use of

3.27 $$(f^i \psi_{,i})^2 \leq \sum_{i,j=1}^{N} f^i f^i \psi_{,j} \psi_{,j}$$

which is easily established by expanding

3.28 $$\sum_{i,j=1}^{N} (f^i \psi_{,j} - f^j \psi_{,i})^2.$$

From the divergence theorem it is clear that

3.29 $$\iiint_V \tilde{\psi}\, \tilde{J}(\tilde{\psi})\, dv = -\iiint_V \tilde{\psi}_{,i}\, a^{ij}\, \tilde{\psi}_{,j}\, dv + \iint_S \tilde{\psi} \frac{\partial \tilde{\psi}}{\partial \nu} ds$$

$$-\iiint_V K z \tilde{\psi}^2 dv - \frac{1}{2} \iint_{D(y_0)} \tilde{\psi}^2 z\, ds + \frac{1}{2} \iint_{D(0)} e^{2Ky_0} G^2 z\, ds$$

$$+\frac{1}{2} \iiint_V \tilde{\psi}^2 \left(\frac{\partial z}{\partial y}\right) dv - \frac{1}{2} \iint_S n_y\, z\, \tilde{\psi}^2 ds.$$

Using 3.7 and 3.20 to write

3.30 $$\tilde{J}(\tilde{\psi}) \leq \tilde{M}_1 |\tilde{\psi}| + \tilde{M}^i |\tilde{\psi}_{,i}| + f(x,y) \exp[K(y_0 - y)]$$

and

3.31 $$\frac{\partial \tilde{\psi}}{\partial \nu} \leq \tilde{M}_2 |\tilde{\psi}| + L(x,y) \exp[K(y_0 - y)],$$

we have by means of the above

3.32 $$\left|\iiint_V \tilde{\varphi}\, \tilde{J}(\tilde{\varphi})\,dv\right| \leq \iiint_V |\tilde{\varphi}|\cdot|\tilde{J}(\tilde{\varphi})|\,dv$$

$$\leq \tilde{M}_1 \iiint_V \tilde{\varphi}^2 dv + \frac{M_3\alpha_2}{2a_0}\iiint_V a^{ij}\tilde{\varphi}_{,i}\tilde{\varphi}_{,j}\,dv + \frac{1}{2\alpha_2}\iiint_V \tilde{\varphi}^2 dv$$

$$+ \frac{\alpha_3}{2}\iiint_V \tilde{\varphi}^2 dv + \frac{1}{2\alpha_3}\iiint_V F^2 \exp[2K(y_0 - y)]\,dv$$

and

3.33 $$\left|\iint_S \tilde{\varphi}\frac{\partial\tilde{\varphi}}{\partial\nu}\,ds\right| \leq \tilde{M}_2\iint_S \tilde{\varphi}^2 ds + \frac{\alpha_4}{2}\iint_S \tilde{\varphi}^2 ds$$

$$+ \frac{1}{2\alpha_4}\iint_S L^2 \exp[2K(y_0 - y)]\,ds$$

where α_2, α_3, and α_4 are arbitrary positive numbers and

3.34 $$M_3 = \sum_1^N (M^i)^2.$$

Substitution of 3.32 and 3.33 into 3.29 yields after the necessary transposition

3.35 $$\left(1 - \frac{M_3\alpha_2}{2a_0}\right)\iiint_V a^{ij}\tilde{\varphi}_{,i}\tilde{\varphi}_{,j}\,dv + \left(Km_2 - \frac{1}{2}M_4 - \tilde{M}_1 - \frac{1}{2\alpha_2} - \frac{\alpha_3}{2}\right)\iiint_V \tilde{\varphi}^2 dv$$

$$\leq \left(\tilde{M}_2 + \frac{\alpha_4}{2} + \frac{1}{2}M_5\right)\iint_S \tilde{\varphi}^2 ds + \frac{1}{2}\iint_{D(0)} G^2 z \exp(2Ky_0)\,ds$$

$$+ \frac{1}{2\alpha_3}\iiint_V F^2 \exp[2K(y_0 - y)]\,dv + \frac{1}{2\alpha_4}\iint_S L^2 \exp[2K(y_0 - y)]\,ds - \frac{1}{2}\iint_{D(y_0)} \tilde{\varphi}^2 z\,ds$$

where

3.36 $$M_4 = \max_V \left\{\frac{\partial z}{\partial y}\right\}$$

and

$$M_5 = \max_{s} \left\{ -n_y z \right\}. \tag{3.37}$$

Except for the first and last integrals, the right side of 3.35 is completely known; the last term may, however, be neglected since z satisfies 3.3. In addition we note that the left side of 3.35 has the same form as the right side of 3.24; and, after a close scrutinization of these inequalities, it is evident that they yield a bound for the integral of the square of $\tilde{\Psi}$ over S provided the arbitrary positive constants α_i and K are properly chosen. In a particular problem the best choice of these arbitrary constants would depend on the functions F, G, and L and the M's appearing above. For definiteness we assume $M_3 \neq 0$ and make the following selections. (If $M_3 = 0$, obvious modifications are made to obtain the same result). Let

$$\alpha_2 = \frac{a_o}{M_3} \tag{3.38}$$

$$\alpha_4 = 1 \tag{3.39}$$

$$\alpha_1 = \frac{a_o m_4}{4M_1} \left(M_2 + \frac{1}{2} + \frac{1}{2} M_5\right)^{-1} \tag{3.40}$$

$$\alpha_3 = 1 \tag{3.41}$$

and

$$K = \frac{1}{m_2} \left\{ \frac{M_4}{2} + \tilde{M}_1 + \frac{M_3}{2a_o} + \frac{1}{2} + (2\tilde{M}_2 + 1 + M_5) \left\langle \frac{2(2\tilde{M}_2 + 1 + M_5) M_1}{a_o m_4^2} + \frac{M_2}{m_4} \right\rangle \right\} = K_1 \tag{3.42}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

where we have assumed that

3.43 $$\tilde{M}_2 + \frac{1}{2} + \frac{1}{2} M_5 > 0.$$

If this is not the case, then one need only choose α_4 so that

3.44 $$\tilde{M}_2 + \frac{\alpha_4}{2} + \frac{1}{2} M_5 > 0$$

and the same result is now obtained by appropriately defining K and α_1. Substitution of the above into 3.35 and 3.24 gives

3.45 $$\iint_S \tilde{\varphi}^2 ds \leq (2\tilde{M}_2 + 1 + M_5)^{-1} \Big\{ \iint_{D(0)} G^2 z \exp(2K_1 y_0) ds$$
$$+ \iiint_V F^2 \exp[2K_1(y_0 - y)] dv + \iint_S L^2 \exp[2K_1(y_0 - y)] ds \Big\}$$

where K_1 is defined by 3.42 (assuming 3.43 is valid). We use the last inequality and 3.20 to obtain if $K_1 \leq 0$

3.46 $$\iint_S \tilde{\varphi}^2 ds \leq (2\tilde{M}_2 + 1 + M_5)^{-1} \Big\{ \iint_{D(0)} G^2 z \, ds$$
$$+ \iiint_V F^2 \exp(-2K_1 y) dv + \iint_S L^2 \exp(-2K_1 y) ds \Big\}$$

and if $K_1 \geq 0$ then

3.47 $$\iint_S \tilde{\varphi}^2 ds \leq (2\tilde{M}_2 + 1 + M_5)^{-1} \Big\{ \iint_{D(0)} G^2 z \exp(2K_1 y_0) ds$$
$$+ \iiint_V F^2 \exp[2K_1(y_0 - y)] dv + \iint_S L^2 \exp[2K_1(y_0 - y)] ds \Big\}.$$

The right side of the applicable inequality is known since F, L and G are square integrable over their domains of definition; hence we may write

$$3.48 \qquad \iint_S \psi^2 \, ds \leq \mathcal{B}$$

where $\mathcal{B}$ is a known number depending on the function φ. After having calculated 3.48, we note that a bound has been obtained for the last term on the right of 3.16.

If a function Γ_p can be determined which is such that the conclusions of Theorem 2.1 are valid and in addition the condition that

$$3.49 \qquad \bar{J}(\Gamma_p) = 0,$$

then it is clear that Γ_p is a fundamental solution of J and the first term on the right of 3.16 is identically zero. Dressel [3] and [4] established the existence of such a solution when the coefficients of the differential operator are sufficiently smooth. From these references it is clear that even when its existence is guaranteed such a function may be quite difficult to construct; hence we shall assume that Γ_p is not known and that the first term on the right of 3.16 must be bounded. To obtain the desired bound we introduce the auxiliary function $\mathcal{F}$ defined by 2.39. Since the differential operator of this section is identical to that of Theorem 2.2, we conclude that α and b may be fixed so that

3.50 $$\mathcal{F} \geq 0 \qquad (x,y) \in V$$

3.51 $$\bar{J}(\mathcal{F}) \leq 0 \qquad (x,y) \in V$$

and

3.52 $$\bar{J}(\mathcal{F}) < 0 \quad , \quad \mathcal{F} > 0$$

for every $\varepsilon > 0$ and y satisfying $0 < \varepsilon \leq y_0 - y \leq y_0$. Next we define the adjoint $\bar{\tilde{J}}$ of $\tilde{J}$ by writing

3.53 $$\bar{\tilde{J}}(\tilde{\varphi}) = \bar{J}(\tilde{\varphi}) - KZ$$

Because the singularity of $\mathcal{F}$ at p is of a slightly lower order than the singularity of the parametrix γ_p, it follows by the proof of Theorem 2.1 that

3.54 $$\lim_{\substack{y \to y_0 \\ y < y_0}} \iint_{D(y)} \mathcal{F} z \tilde{\varphi}^2 \, ds$$

$$= 0$$

since Z is strictly positive and $\bar{\alpha}$ is a non-negative number. By means of Green's Theorem we write

3.55 $$\lim_{y \to y_0} \iiint_V \{ \mathcal{F} \tilde{J}(\tilde{\varphi}^2) - \tilde{\varphi}^2 \bar{\tilde{J}}(\mathcal{F}) \} \, dv$$

$$= \iint_S \{ \mathcal{F} \frac{\partial \tilde{\varphi}^2}{\partial \nu} - \tilde{\varphi}^2 \frac{\partial \mathcal{F}}{\partial \nu} \} \, ds - \iint_S n_y \mathcal{F} z \tilde{\varphi}^2 \, ds$$

$$+ \iint_{D(0)} \mathcal{F} z \tilde{\varphi}^2 \, ds$$

where we have used 3.54 and the fact that

3.56
$$\lim_{\substack{y \to y_0 \\ y < y_0}} \iint_{S(y)} \left\{ \mathcal{F} \frac{\partial \tilde{\varphi}^2}{\partial \nu} - \tilde{\varphi}^2 \frac{\partial \mathcal{F}}{\partial \nu} \right\} ds$$

$$= \iint_S \left\{ \mathcal{F} \frac{\partial \tilde{\varphi}^2}{\partial \nu} - \tilde{\varphi}^2 \frac{\partial \mathcal{F}}{\partial \nu} \right\} ds$$

for $p \in D(y_0)$. If we assume that K is non-negative, then expansion of 3.55 yields

3.57
$$\lim_{\substack{y \to y_0 \\ y < y_0}} \iiint_{V(y)} \left\{ 2\mathcal{F} \tilde{\varphi} \tilde{J}(\tilde{\varphi}) - \tilde{\varphi}^2 \bar{J}(\mathcal{F}) \right.$$

$$\left. + 2\mathcal{F} \tilde{\varphi}_{,i} a^{ij} \tilde{\varphi}_{,j} + 2 K z \mathcal{F} \tilde{\varphi}^2 \right\} dv$$

$$\leq \alpha_1 \iint_S (\mathcal{F} L)^2 \exp[2K(y_0 - y)] ds + \frac{1}{\alpha_1} \iint_S \tilde{\varphi}^2 ds$$

$$+ 2\tilde{M}_2 \iint_S \mathcal{F} \tilde{\varphi}^2 ds - \iint_S \tilde{\varphi}^2 \frac{\partial \mathcal{F}}{\partial \nu} ds - \iint_S m_y \mathcal{F} z \tilde{\varphi}^2 ds$$

$$+ \iint_{D(0)} \mathcal{F} G^2 z \exp(2Ky_0) ds$$

$$\leq \iint_S (\mathcal{F} L)^2 \exp[2K(y_0 - y)] ds + M_6 \mathcal{B} \exp(2Ky_0)$$

$$+ \iint_{D(0)} \mathcal{F} G^2 z \exp(2Ky_0) ds = \mathcal{J}$$

where in the right inequality we have for definiteness set $\alpha_1 = 1$ and

[illegible]

3.58 $$M_6 = \max_s \left\{0, 1+2\tilde{M}_2\mathcal{F} - \frac{\partial\mathcal{F}}{\partial\nu} - n_y\mathcal{F}z\right\}.$$

By means of 3.7 and 3.21 we have

3.59 $$\iiint\limits_{V(y)} 2\mathcal{F}\tilde{\varphi}\tilde{J}(\tilde{\varphi})\,dV = 2\iiint\limits_{V(y)} \mathcal{F}\tilde{\varphi}F\exp[K(y_0-y)]\,dV$$

$$+2\iiint\limits_{V(y)}\mathcal{F}\tilde{\varphi}\{f(x,y,w,\nabla w)-f(x,y,\varphi,\nabla\varphi)\}\exp[K(y_0-y)]\,dV.$$

Also

3.60 $$2\iiint\limits_{V(y)}\mathcal{F}|\tilde{\varphi}|\cdot|f(x,y,w,\nabla w)$$

$$-f(x,y,\varphi,\nabla\varphi)|\exp[K(y_0-y)]\,dV$$

$$\leq 2\tilde{M}_1\iiint\limits_{V(y)}\mathcal{F}\tilde{\varphi}^2\,dV + 2\tilde{M}^i\iiint\limits_{V(y)}\mathcal{F}|\tilde{\varphi}|\cdot|\tilde{\varphi}_{,i}|\,dV$$

$$\leq (2\tilde{M}_1+\alpha_5)\iiint\limits_{V(y)}\mathcal{F}\tilde{\varphi}^2\,dV + \frac{M_3}{\alpha_5 a_0}\iiint\limits_{V(y)}\mathcal{F}a^{ij}\tilde{\varphi}_{,i}\tilde{\varphi}_{,j}\,dV$$

where M_3 is given by 3.34 and α_5 is an arbitrary positive number. Let

3.61 $$\alpha_5 = \frac{M_3}{2a_0}$$

then by choosing K so that

3.62 $$K = \frac{1}{2m_2}\left\{2\tilde{M}_1 + \frac{M_3}{2a_0}\right\} = K_2$$

and using 3.60, it is evident that 3.57 may be rewritten as

3.63 $$\lim_{y\to y_0}\iiint\limits_{V(y)}\{2\mathcal{F}\tilde{\varphi}F\exp[K_2(y_0-y)] - \tilde{\varphi}^2\tilde{J}(\mathcal{F})\}\,dV \leq J_2.$$

With $\mathcal{F}$ defined on $D(y_0) - \{p\}$ as is indicated in Chapter II and with the constants δ_2 and ρ_2 chosen so that $\rho_2 \leq \rho_0$ and $\delta_2 \leq \delta_0$, we have from 2.35 for appropriately defined constants c_i, $i = 10, \ldots 13$,

3.64 $$\lim_{y \to y_0} \iiint_{V-V(y)} \{2\mathcal{F}\tilde{\varphi} F \exp[K_2(y_0 - y)] - \tilde{\varphi}^2 J(\mathcal{F})\} dV$$

$$= \lim_{y \to y_0} \iiint_{V(\rho_2, \delta_2)-V(y)} \{2\mathcal{F}\tilde{\varphi}[f(x, y, \varphi, \nabla\varphi)$$

$$- J(\varphi)] \exp[K_2(y_0 - y)] - \tilde{\varphi}^2 J(\mathcal{F})\} dV$$

$$= \lim_{y \to y_0} \iiint_{V(\rho_2, \delta_2)-V(y)} \{c_{10}\mathcal{F} + [c_{11}(y_0 - y)^{-\frac{N+1}{2}} + c_{12}(y_0 - y)^{-\frac{N+3}{2}} r^2$$

$$+ c_{13}(y_0 - y)^{-\frac{N-1}{2}}] \exp\langle -[8a_1 N(y_0 - y)]^{-1} [(2N-1) z(p) r^2] \rangle\} dV$$

$$= 0$$

In arriving at the last result we have integrated by parts first with respect to r and then with respect to y (see the proof of part a of Theorem 2.1). From the above it follows that

3.65 $$\iiint_V \{2\mathcal{F}\tilde{\varphi} F \exp[K_2(y_0 - y)] - \tilde{\varphi}^2 J(\mathcal{F})\} dV \leq \mathcal{J}_2 .$$

From Schwarz's inequality and the fact that $\mathcal{F}$ satisfies 3.51 it follows that

$$3.66 \qquad -\left|\iiint_V \mathcal{F}\,\tilde{\varphi}\,F \exp\left[K_2(y_0-y)\right]dv\right|$$

$$\geq -\left\{-\iiint_V \tilde{\varphi}^2\,\bar{J}(\mathcal{F})\,dv\right\}^{1/2}\cdot$$

$$\cdot\left\{\iiint_V \left[-\bar{J}(\mathcal{F})\right]^{-1}(\mathcal{F}F)^2 \exp\left[2K_2(y_0-y)\right]dv\right\}^{\frac{1}{2}}.$$

The factors on the right of 3.66 each exist. The first exists since $\tilde{\varphi}$ is continuous on $\bar{V}$ and is hence bounded on V while $\bar{J}(\mathcal{F})$ is easily seen to be integrable over V. For the second factor we note that the singularity of $\mathcal{F}^2$ at $p \in D(y_0)$ is less than that of $J(\hat{\gamma}p)^2$ and the result may thus be obtained directly from Theorem 2.1. Due to the above

$$3.67 \qquad \left\{\iiint_V -\bar{J}(\mathcal{F})\,\tilde{\varphi}^2 dv\right\}^{\frac{1}{2}} \leq \left\{\iiint_V \left[-\bar{J}(\mathcal{F})\right]^{-1}(\mathcal{F}F)^2 \exp\left[2K_2(y_0-y)\right]dv\right\}^{\frac{1}{2}}$$

$$+\left\{-\iiint \tilde{\varphi}^2\,\bar{J}(\mathcal{F})\,dv + 2\iiint_V \mathcal{F}\,\tilde{\varphi}\,F \exp\left[K_2(y_0-y)\right]dv\right.$$

$$\left.+\iiint_V \left[-\bar{J}(\mathcal{F})\right]^{-1}(\mathcal{F}F)^2 \exp\left[2K_2(y_0-y)\right]dv\right\}^{\frac{1}{2}}$$

and then, after substitution of 3.65 and 3.20, we write

$$3.68 \qquad \left\{-\iiint_V \bar{J}(\mathcal{F})\,\psi^2 dv\right\}^{1/2}$$

$$\leq \left\{\iiint_V \left[-\bar{J}(\mathcal{F})\right]^{-1}(\mathcal{F}F)^2 \exp\left[2K_2(y_0-y)\right]dv\right\}^{1/2}$$

$$+\left\{\mathcal{J}_2 + \iiint_V \left[-\bar{J}(\mathcal{F})\right]^{-1}(\mathcal{F}F)^2 \exp\left[2K_2(y_0-y)\right]dv\right\}^{1/2}.$$

Because of the form of $\mathcal{F}$ and γ_p, we use Schwarz's inequality to obtain

3.69
$$\left|\lim_{y\to y_0} \iiint_{V(y)} \psi\, \bar{J}(\gamma_p)\, dV\right|$$
$$\leq \left\{\iiint_V [-\bar{J}(\mathcal{F})]\, \bar{J}(\gamma_p)^2 dV\right\}^{\frac{1}{2}} \cdot$$
$$\cdot \left\langle \left\{\iiint_V [-\bar{J}(\mathcal{F})]^{-1} (\mathcal{F} F)^2 \exp[2K_2(y_0 - y)]\, dV\right\}^{\frac{1}{2}}\right.$$
$$\left. + \left\{g_2 + \iiint_V [-\bar{J}(\mathcal{F})]^{-1} (\mathcal{F} F)^2 \exp[2K_2(y_0 - y)]\, dV\right\}^{1/2}\right\rangle$$

where the first factor on the left exists due to Theorem 2.4 and the second factor by the remark following inequality 3.66. It is clear that 3.69 is the desired bound for the first term on the right of 3.16.

To obtain the desired bound it is now only necessary to bound the second term on the right of 3.16. We write

3.70
$$\left|\lim_{y\to y_0} \iiint_{V(y)} \gamma_p \{f(x,y,w,\nabla w) - f(x,y,\varphi,\nabla\varphi)\}\, dV\right|$$
$$\leq \lim_{y\to y_0} \iiint_{V(y)} \gamma_p \{\tilde{M}_1 |\psi| + \tilde{M}^i |\psi_{,i}|\}\, dV$$
$$\leq \left\{\iiint_V \gamma_p^2 \mathcal{F}^{-1} \exp[-2K(y_0-y)]\, dV\right\}^{1/2} \left\{\lim_{y\to y_0} \iiint_{V(y)} \mathcal{F}\, [\tilde{M}_1 |\Psi| + \tilde{M}^i |\Psi_{,i}|]^2 dV\right\}^{1/2}$$
$$\leq \left\{\iiint_V \gamma_p^2 \mathcal{F}^{-1} \exp[-2K(y_0-y)]\, dV\right\}^{1/2} \left\{\lim_{y\to y_0} \iiint_{V(y)} [2\tilde{M}_1^2 \mathcal{F} \Psi^2 + 2M_3 a_0^{-1} \mathcal{F} \Psi_{,i}\, a^{ij} \Psi_{,j}]\, dV\right\}^{1/2}.$$

In the last inequality the first factor on the right exists since for c_{14} appropriately defined

3.71
$$\iiint_V \gamma_p^2 \mathcal{F}^{-1} \exp\left[2K(y_0-y)\right] dv \leq C_{14} +$$

$$+C_{14}\int_{y_0-\delta_0}^{y_0} \int_{\omega_N} \int_0^{\varepsilon} (y_0-y)^{-\frac{N+1}{2}} \exp\left\{-\left[8Na_1(y_0-y)\right]^{-1}\left[(2N-1)z(p)r^2\right]\right\} r^{N-1}\, dr\, d\omega\, dy$$

for ε sufficiently small. The integral in the above expression is seen to exist by means of integration by parts as in the proof of part (b) of Theorem 2.1. The last factor on the right of 3.70 is now bounded by reconsidering 3.57 and 3.59. From these inequalities we obtain

3.72
$$J \geq \lim_{\substack{y \to y_0 \\ y < y_0}} \iiint_{V(y)} \left\{2\mathcal{F}\tilde{\varphi}\tilde{J}(\tilde{\varphi}) - \tilde{\varphi}^2 \tilde{J}(\mathcal{F}) + 2\mathcal{F}\tilde{\varphi}_{,i} a^{ij} \tilde{\varphi}_{,j} + 2Kz\mathcal{F}\tilde{\varphi}^2\right\} dv$$

$$\geq \lim_{\substack{y \to y_0 \\ y < y_0}} \iiint_{V(y)} \left\{-\frac{1}{\alpha_1}\mathcal{F} F^2 \exp\left[2K(y_0-y)\right]\right.$$

$$\left. + (2Kz - \alpha_1 - 2\hat{M}_1 - \alpha_2)\mathcal{F}\tilde{\varphi}^2 + \left(2 - \frac{M_3}{\alpha_2 a_0}\right)\mathcal{F} a^{ij} \tilde{\varphi}_{,i} \tilde{\varphi}_{,j}\right\} dv.$$

In the last equation the choice of the α_i's depend on the magnitude of $M_3 a_o^{-1}$ and the boundary data. For definiteness we set

3.73 $\quad \alpha_1 = 1$

and

3.73 a $\quad \alpha_2 = \frac{1}{2} M_3 (a_o - M_3)^{-1} \qquad$ for $\frac{1}{2} \geq M_3 a_o^{-1}$

or

3.73 b $$\alpha_2 = M_3 a_o^{-1} \qquad \text{for } M_3 a_o^{-1} > \tfrac{1}{2}.$$

Next we choose

3.74 a $$K = \frac{1}{2m_2}\left\{1+2\tilde{M}_1+\tfrac{1}{2}M_3(a_o-M_3)^{-1}+2\tilde{M}_1^{\,2}\right\} = K_3$$

or

3.74 b $$K = \frac{1}{2m_2}\left\{1+2\tilde{M}_1+M_3 a_o^{-1}+a_o M_3^{-1}\tilde{M}_1^{\,2}\right\} = K_3$$

whence either

3.75 a $$\lim_{y\to y_0} \iiint_{V(y)} \left[2\tilde{M}_1^{\,2}\mathcal{F}\tilde{\varphi}^2 + 2M_3 a_0^{-1}\mathcal{F} a^{ij}\tilde{\varphi}_{,i}\,\tilde{\varphi}_{,j}\right] dV$$

$$\leq \mathcal{J}_3 + \iiint_V \mathcal{F} F^2 \exp\left[2K_3(y_0-y)\right] dV = \tilde{\mathcal{J}}$$

or

3.75 b $$\lim_{y\to y_0} \iiint_{V(y)} \left[2\tilde{M}_1\mathcal{F}\tilde{\varphi}^2 + 2M_3 a_0^{-1}\mathcal{F} a^{ij}\tilde{\varphi}_{,i}\tilde{\varphi}_{,j}\right] dV$$

$$\leq (2M_3 a_0^{-1})\left\{\mathcal{J}_3 + \iiint_V \mathcal{F} F^2 \exp\left[2K_3(y_0-y)\right] dV\right\} = \tilde{\mathcal{J}}$$

where $\mathcal{J}_3$ is given by 3.57 with $K = K_3$.

Subject to the remark following 3.71 it is clear that equation 3.70 and the proper form of 3.75 provide a bound for the second term on the right of 3.16.

By combining the results of this section, the solution of 3.1 is bounded at the point $p \in D(y_o)$ by

[illegible]

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[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

3.75.5
$$|w(x_0,y_0)-\chi(p)|$$
$$\leq \left\{ \iiint_V [-\bar{J}(\mathcal{F})]^{-1} \bar{J}(\gamma_p)^2 dV \right\}^{1/2} \cdot$$
$$\cdot \left\langle \left\{ \iiint_V [-\bar{J}(\mathcal{F})] (\mathcal{F} F)^2 \exp\left[2K_2(y_0-y)\right] dV \right\}^{1/2} \right.$$
$$\left. + \left\{ \mathcal{J}_2 + \iiint_V [-\bar{J}(\mathcal{F})]^{-1} (\mathcal{F} F)^2 \exp\left[2K_2(y_0-y)\right] dV \right\}^{1/2} \right\rangle$$
$$+ \left\{ \iiint_V \gamma_p^2 \mathcal{F}^{-1} \exp\left[-2K_3(y_0-y)\right] dV \right\}^{1/2} \left\{ \tilde{\mathcal{J}} \right\}^{1/2}$$
$$\mathcal{B}^{1/2} \left\langle \left\{ \iint_S \left(\frac{\partial \gamma_p}{\partial \nu} + n_y z \gamma_p \right)^2 ds \right\}^{1/2} + \hat{M}_2 \left\{ \iint_S \gamma_p^2 ds \right\}^{1/2} \right\rangle .$$

In the last inequality the functions γ_p and $\mathcal{F}$ are given by 2.10 and 2.39, and the constants K_i by 3.42, 3.62, and 3.74. The computable bounds $\mathcal{B}$, $\mathcal{J}_2$ and $\tilde{\mathcal{J}}$ are given respectively by 3.48, 3.57, and 3.75 while the function χ(p) is defined by 3.17.

If problem 3.1 admits a solution w which is twice piecewise continuously differentiable in y for $(x,y) \in \bar{V}$, then theoretically the pointwise bound may be made arbitrarily small by improvement of the approximation function φ. In this regard we note that the right side of 3.75.5 is zero if F = L = G = 0. Furthermore, due to the form of the bound, it is clear that the result may be improved by means of the Rayleigh-Ritz technique provided the problem is linear.

By referring to the proofs of the appropriate theorems of Chapter II and the methods of this section, one observes that the boundary value problem given in 3.1 may be generalized by allowing the components $a^{ij} = a^{ij}(x,y)$ to be non-symmetric. Such a problem was not considered since, even though the techniques are exactly the same, the individual inequality expressions are naturally much more involved. In the next section, where the matrix a^{ij} is required to be symmetric, a method is given for symmetrizing a general problem.

As a second generalization it is easy to see that the differentiability of the a^{ij}, $i, j = 1, 2 \ldots N$ may be reduced by merely requiring that $a,{}_i^{ij}$ and $a,{}_j^{ij}$ be continuous in the interior of a finite number of regions the sum of which is $D(y)$ for $o < y < y_o$. The a^{ij} are themselves continuous in $D(y)$ for $o \leq y \leq y_o$ and piecewise continuous in $\bar{V}$. Under this reduced differentiability, one may easily show that the techniques of this section are valid.

2. The Non-Linear Dirichlet Problem

Our aim is to bound the solution of a normally parabolic problem which is such that the solution is initially given and thereafter satisfies a non-linear normally parabolic differential equation in the interior while assuming given values of the boundary.

We assume that V is an $N + 1$ dimensional domain with boundary $D(y_o) \cup \bar{S} \cup D(o)$ which is such that 2.4 and 2.5 are

satisfied. The boundary value problem may be stated explicitly as follows:

$$3.76 \qquad \begin{cases} J(w) = (a^{ij}\, w_{,i})_{,j} - z\dfrac{\partial w}{\partial y} = f(x,y,w,\nabla w) & (x,y) \in V \\ & -\infty < w, \nabla w < \infty \\ w(x,o) = g(x) & (x,y) \in D(o) \\ w(x,y) = h(x,y) & (x,y) \in S \end{cases}$$

where w is continuous in $V \cup D(y_o)$, continuously differentiable in $V \cup S$ and is such that its second derivative in x and mixed derivative in x and y are continuous in the interior of a finite number of subregions, the sum of which is D(y) for every y satisfying $o < y < y_o$. In addition it is necessary to assume that

$$3.77 \qquad \begin{cases} \lim\limits_{\substack{(x,y) \to (o,x) \\ (x,y) \in V}} w(x,y) = g(x) & \\ \lim\limits_{\substack{(x,y) \to (o,x) \\ (x,y) \in V}} w_{,i}(x,y) = g_i(x) & i = 1, 2 \ldots N. \end{cases}$$

The differentiability conditions which the solution of the present problem is to satisfy are obviously much stronger than those which were required for the solution of the boundary value problem considered in the first section of this chapter. Additional conditions must also be placed in the coefficients of the differential operator; in particular, the components of the symmetric matrix $a^{ij} = a^{ij}(x,y)$ are piecewise continuously differentiable in $\overline{V}$ and satisfy 2.7 while the function $z = z(x,y)$ is piecewise continuously differentiable in y for $(x,y) \in \overline{V}$ and satisfies

3.78 $$\min_{\overline{V}} z(x,y) = m_2 > 0.$$

The boundary data f, g, and $g_{,i}$ are assumed to be integrable and square integrable over their respective domains of definition for any bounded continuous function w which is continuously differentiable in x for $(x,y) \in \overline{V}$. Moreover, the function f satisfies a Lipschitz condition in all but its first two arguments. Hence there exist positive numbers $\widetilde{M}_1$ and $\widetilde{M}^i$, i = 1, 2 . . . N, such that

3.79 $$|f(x,y,w_1,\nabla w_1)-f(x,y,w_2,\nabla w_2)| \leq \widetilde{M}_1 |w_1-w_2| + \widetilde{M}^i |w_{1,i}-w_{2,i}|$$
$$\text{for } (x,y) \in V, -\infty < w, \nabla x < \infty.$$

We remark that the problem defined above and the additional problem which we shall obtain from it by introducing an arbitrary function are very similar to a problem for which Friedman, [10] and [12], and Zeragiya, [27] and [28] established the existence of a solution.

The desired bound for the solution function will be obtained at the point $p \in D(y_o)$. As in the previous section we assume that z satisfies 2.9 at p (notice that 2.8 is satisfied due to the differentiability of the a^{ij}) and that f is bounded in some neighborhood of p which is contained in $\overline{V}$ for any continuous function w which is continuously differentiable in x for $(x,y) \in \overline{V}$, that is, we suppose that 3.5 is satisfied. To obtain the desired result we choose an arbitrary function $\varphi = \varphi(x,y)$ which is such that $J(\varphi)$ and $\varphi(x,y)$ approximate $f(x,y,\varphi,\nabla\varphi)$, g(x) and h(x,y)

respectively. In order that the divergence theorem may be applied in what follows we further require that φ be twice piecewise continuously differentiable in x and piecewise continuously differentiable in y for $(x,y) \in \overline{V}$. Also we require that the $\varphi,_i$ be piecewise continuously differentiable in y for $(x,y) \in \overline{V}$.

Following the techniques of the first section of this chapter we define

$$3.80 \qquad \psi(x,y) = w(x,y) - \varphi(x,y)$$

and then compute

$$3.81 \qquad \begin{cases} J(\psi) = f(x,y,w,\nabla w) - f(x,y,\varphi,\nabla\varphi) + F(x,y) & (x,y) \in V \\ \psi(x,o) = g(x) - \varphi(x,o) = G(x) & (x,y) \in D(o) \\ \psi(x,y) = h(x,y) - \varphi(x,y) = H(x,y) & (x,y) \in S \end{cases}$$

where

$$3.82 \qquad F(x,y) = f(x,y,\varphi,\nabla\varphi) - J(\varphi).$$

For later use we note that

$$3.83 \qquad |J(\psi)| \leq \widetilde{M}_1 \, |\psi| + \widetilde{M}^i \, |\psi_{,i}| + |F(x,y)|.$$

With $\overline{J}(\psi)$ and γ_p defined respectively by 2.6 and 2.10 we note that 3.15 is valid for the present problem. Substitution of 3.81, 3.82, and 3.83 into that equation then yields after transposition

3.84 $$\left|w(x_0,y_0)-\chi(p)\right| \le \left|\lim_{\substack{y\to y_0\\ y<y_0}} \iiint_{V(y)} \psi\, \bar{J}(\gamma_p)\,dv\right|$$

$$+\left|\lim_{\substack{y\to y_0\\ y<y_0}} \iiint_{V(y)} \gamma_p \left\{ f(x,y,w,\nabla w) - f(x,y,\varphi,\nabla\varphi)\right\} dv\right|$$

$$+\left\{\iint_S \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds\right\}^{1/2} \left\{\iint_S \gamma_p^2\, ds\right\}^{1/2}$$

where

3.85 $$\chi(p) = \varphi(x_0,y_0) - \iint_S H \frac{\partial\gamma_p}{\partial\nu}\,ds - \lim_{\substack{y\to y_0\\ y<y_0}} \iiint_{V(y)} \gamma_p F\,dv$$

$$+\iint_{D(0)} \gamma_p z G\,ds - \iint_S n_y \gamma_p z H\,ds.$$

The volume integral on the right of 3.85 is proved to exist by the argument associated with equations 3.18 and 3.19. Hence $w(x_0,y_0)$ is the only unknown term of the left side of 3.84 and our desired error bound is then clearly available as soon as the expressions on the right of 3.84 are bounded. Since the unknown volume integrals will be bounded in terms of known quantities and the integral of the square of the conormal derivative of ψ over S, the last term on the right of 3.84 is considered first. For this purpose use is made of 3.81 to rewrite 2.113 as

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]

3.86 $$\iint_S \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds$$

$$\leq \mathcal{H} + \frac{2c_1}{m_3} \iiint_V a^{ij} \psi_{,i} \psi_{,j} \, dV$$

$$+ \frac{4}{m_3} \iiint_V f^k \psi_{,k} \left\{ z \frac{\partial \psi}{\partial y} + f(x,y,w,\nabla w) - f(x,y,\varphi,\nabla\varphi) + F(x,y) \right\} dV$$

where $\mathcal{H}$ is given by 2.114. Since f satisfies 3.79 we may write

3.87 $$\iiint_V f^k \psi_{,k} \left\{ z \frac{\partial \psi}{\partial y} + f(x,y,w,\nabla w) - f(x,y,\varphi,\nabla\varphi) + F(x,y) \right\} dV$$

$$\leq \frac{1}{2\alpha_1} \iiint_V z^2 (f^k \psi_{,k})^2 dV + \frac{\alpha_1}{2} \iiint_V \left(\frac{\partial \psi}{\partial y}\right)^2 dV + \frac{1}{2} \iiint_V (f^k \psi_{,k})^2 dV$$

$$+ \frac{\tilde{M}_1^2}{2} \iiint_V \psi^2 dV + \frac{1}{2} \iiint_V (f^k \psi_{,k})^2 dV + \frac{1}{2} \iiint_V (\tilde{M}^i \psi_{,i})^2 dV$$

$$+ \frac{1}{2} \iiint_V (f^k \psi_{,k})^2 dV + \frac{1}{2} \iiint_V F^2 dV$$

$$\leq \left(\frac{M_7}{2\alpha_1 a_0} + \frac{3M_1}{2a_0} + \frac{M_9}{2a_0} \right) \iiint_V a^{ij} \psi_{,i} \psi_{,j} \, dV + \frac{\tilde{M}_1^2}{2} \iiint_V \psi^2 dV$$

$$+ \frac{\alpha_1}{2} \iiint_V \left(\frac{\partial \psi}{\partial y}\right)^2 dV + \frac{1}{2} \iiint_V F^2 dV$$

where

3.88 $$M_7 = \max_V \left\{ z^2 \sum_1^N (f^i)^2 \right\}$$

and α_1 is any positive number. In the last inequality the constants M_1 and M_3 are given by 3.25 and 3.34 respectively. Hence 3.86 becomes

3.89
$$\iint_S \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds \leq \mathcal{H} + \frac{2}{m_3}\left(c_1 + \frac{M_7}{\alpha_1 a_0} + \frac{3M_1}{a_0} + \frac{M_3}{a_0}\right)\iiint_V a^{ij}\psi_{,i}\psi_{,j}\,dV$$
$$+\frac{2\tilde{M}_1^2}{m_3}\iiint_V \psi^2 dV + \frac{2\alpha_1}{m_3}\iiint_V\left(\frac{\partial\psi}{\partial y}\right)^2 dV + \frac{2}{m_3}\iiint_V F^2 dV.$$

Due to the differentiability of the solution of 3.81, we may write by means of the divergence theorem

3.90
$$\iiint_V \frac{\partial\psi}{\partial y} J(\psi)\,dV \leq -\iiint_V \frac{\partial\psi_{,j}}{\partial y} a^{ij}\psi_{,i}\,dV$$
$$+\iint_S \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial\nu}\,ds - m_2\iiint_V\left(\frac{\partial\psi}{\partial y}\right)^2 dV.$$

Application of 3.83 to the left side of the last inequality yields

3.91
$$\left|\iiint_V \frac{\partial\psi}{\partial y} J(\psi)\,dV\right| \leq \iiint_V\left|\frac{\partial\psi}{\partial y}\right|\cdot\left\{\tilde{M}_1|\psi| + \tilde{M}^i|\psi_{,i}| + |F|\right\}dV$$
$$\leq \frac{\varepsilon_1}{2}\iiint_V\left(\frac{\partial\psi}{\partial y}\right)^2 dV + \frac{\tilde{M}_1}{2\varepsilon_1}\iiint_V\psi^2 dV + \frac{\varepsilon_2}{2}\iiint_V\left(\frac{\partial\psi}{\partial y}\right)^2 dV$$
$$+\frac{M_3}{2\varepsilon_2 a_0}\iiint_V a^{ij}\psi_{,i}\psi_{,j}\,dV + \frac{\varepsilon_3}{2}\iiint_V\left(\frac{\partial\psi}{\partial y}\right)^2 dV + \frac{1}{2\varepsilon_3}\iiint_V F^2 dV$$

where the ε_i are arbitrary positive numbers. Subsequently, when we use 3.91 to rewrite 3.90, the arbitrary constants will for definiteness be defined as follows:

3.92 $$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{m_2}{3}$$

The first term on the right of 3.90 is bounded by noting that

3.93 $$-\iiint_V \frac{\partial \psi_{,j}}{\partial y} a^{ij} \psi_{,i}\, dv = -\frac{1}{2} \lim_{y \to y_0} \iint_{D(y)} \psi_{,j} a^{ij} \psi_{,i}\, ds$$

$$+\frac{1}{2} \iint_{D(0)} G_{,j} a^{ij} G_{,i}\, ds + \frac{M_8}{2a_0} \iiint_V \psi_{,i} a^{ij} \psi_{,j}\, dv$$

$$-\frac{1}{2} \iint_S n_y a^{ij} \psi_{,i} \psi_{,j}\, ds$$

where

3.94 $$M_8 = \max_V \left\{ \sum_{ij}^{N} \frac{\partial a^{ij}}{\partial y} \frac{\partial a^{ij}}{\partial y} \right\}^{\frac{1}{2}}.$$

Using 2.102 the last term on the right of 3.93 may be rewritten and bounded by

3.95 $$-\frac{1}{2} \iint_S n_y a^{ij} \psi_{,i} \psi_{,j}\, ds = -\frac{1}{2} \iint_S n^{-1} n_y \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds - \frac{1}{2} \iint_S n^{-1} n_y \left(\frac{\partial \psi}{\partial t}\right)^2 ds$$

$$\leq \frac{M_9}{2} \iint_S \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds - \frac{1}{2} \iint_S n^{-1} n_y \left(\frac{\partial \psi}{\partial t}\right)^2 ds$$

where

3.96 $$M_9 = \max_S \left\{ -n^{-1} n_y \right\}.$$

To estimate the second term on the right of 3.91 we first write for an arbitrary positive number α_2

3.97 $$\iint_S \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial \nu} ds \leq \frac{\alpha_2}{2} \iint_S \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds + \frac{1}{2\alpha_2} \iint_S \left(\frac{\partial \psi}{\partial y}\right)^2 ds$$

and then, almost everywhere on S, define the vector

3.98 $$T = (T^1, \ldots T^N, T^y)$$

where T is tangent to the surface S, is orthogonal to the vector (t^i, o), given by 2.97, and has unit Euclidean length, i.e.

3.99 $$\sum_1^N (T^i)^2 + T^{y^2} = 1$$

We denote a directional derivative in the T direction by

3.100 $$\frac{\partial \psi}{\partial T} = \psi_{,i} T^i + \frac{\partial \psi}{\partial y} T^y.$$

Since the space is Euclidean,

3.101 $$n^i = n_i \qquad \text{for every } i$$

whence

3.102 $$\frac{\partial \psi}{\partial y} = n_y \left[\sum_1^N \psi_{,i} n_i + \frac{\partial \psi}{\partial y} n_y \right] + (1 - n_y^2)^{\frac{1}{2}} \frac{\partial \psi}{\partial T}.$$

Transposing and squaring 3.102 yields

3.103 $$(1-n_y^2)^2\left(\frac{\partial\psi}{\partial y}\right)^2 = \left\{n_y\sum_1^N \psi_{,i}n_i + (1-n_y^2)^{\frac{1}{2}}\frac{\partial\psi}{\partial T}\right\}^2$$

$$\leq 2n_y^2\left(\sum_1^N n_n^2\right)a_0^{-1}a^{ij}\psi_{,i}\psi_{,j} + 2(1-n_y^2)\left(\frac{\partial\psi}{\partial T}\right)^2$$

$$= 2n_y^2\left(\sum_1^N n_n^2\right)(a_0n)^{-1}\left(\frac{\partial\psi}{\partial\nu}\right)^2 + 2n_y^2\left(\sum_1^N n_n^2\right)(a_0n)^{-1}\left(\frac{\partial\psi}{\partial t}\right)^2$$

$$+ 2(1-n_y^2)\left(\frac{\partial\psi}{\partial T}\right)^2.$$

Since 2.5 is satisfied on S,

3.104 $$1 - n_y^2 > 0$$

and hence from 3.103

3.105 $$\iint_S\left(\frac{\partial\psi}{\partial y}\right)^2 ds \leq M_{10}\iint_S\left(\frac{\partial\psi}{\partial\nu}\right)^2 ds + \iint_S 2\left[1-n_y^2\right]^{-1}\left(\frac{\partial\psi}{\partial T}\right)^2 ds$$

$$+ \iint_S 2n_y^2\left[a_0n(1-n_y^2)\right]^{-1}\left(\frac{\partial\psi}{\partial t}\right)^2 ds$$

where

3.106 $$M_{10} = \max_S\left\{2n_y^2\left[a_0n(1-n_y^2)\right]^{-1}\right\}.$$

By applying the above results to 3.90 we have

3.107
$$\frac{m_2}{2}\iiint_V \left(\frac{\partial\psi}{\partial y}\right)^2 dv \leq \frac{3\tilde{M}_1}{2m_2}\iiint_V \psi^2 dv$$

$$+\left(\frac{3M_3}{2m_2a_0}+\frac{M_8}{2a_0}\right)\iiint_V a^{ij}\psi_{,i}\psi_{,j}\,dv$$

$$+\left(\frac{M_9}{2}+\frac{\alpha_2}{2}+\frac{M_{10}}{2\alpha_2}\right)\iint_S \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds+\frac{3}{2m_2}\iiint_V F^2 dv$$

$$+\frac{1}{2}\iint_{D(0)} G_{,j}\,a^{ij}G_{,i}\,ds-\frac{1}{2}\iint_S n^{-1}n_y\left(\frac{\partial\psi}{\partial t}\right)^2 ds$$

$$+\frac{1}{\alpha_2}\iint_S n_y^2\left[a_0 n\left(1-n_y^{\,2}\right)\right]^{-1}\left(\frac{\partial\psi}{\partial t}\right)^2 ds$$

$$+\frac{1}{\alpha_1}\iint_S \left(1-n_y^{\,2}\right)^{-1}\left(\frac{\partial\psi}{\partial T}\right)^2 ds-\frac{1}{2}\lim_{\substack{y\to y_0\\ y<y_0}}\iint_{D(y)}\psi_{,i}\,a^{ij}\psi_{,j}\,ds.$$

In 3.107, let

3.108 $\alpha_2 = \sqrt{M_{10}}$

which minimizes the coefficient of the integral of the square of the conormal derivative. For what follows we assume that $(M_9+2\sqrt{M_{10}})$ is a strictly positive number; when this is not the case, obvious modifications are made to obtain the same result. We set

3.109 $\alpha_1 = \frac{m_2m_3}{4}\,(M_9+2\sqrt{M_{10}})^{-1}$

and then rewrite 3.89 as

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3.110 $$\iint\limits_S \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds \leq 2\mathcal{H} + \frac{4}{m_3}\Big\{ C_1 + \frac{4M_7}{a_0 m_2 m_3}\left(M_9 + 2\sqrt{M_{10}}\right)$$

$$+\frac{3M_1}{a_0} + \frac{M_3}{a_0} + \frac{m_3}{2}\left(M_9 + 2\sqrt{M_{10}}\right)^{-1}\left(\frac{3M_3}{2m_2 a_0} + \frac{M_8}{2a_0}\right)\Big\} \iiint\limits_V a^{ij}\psi_{,i}\psi_{,j}\, dV$$

$$+2\left\{\frac{2\tilde{M}_1^2}{m_3} + \frac{3\tilde{M}_1}{2m_2}\left(M_9 + 2\sqrt{M_{10}}\right)^{-1}\right\}\iiint\limits_V \psi^2 dV + \frac{4}{m_3}\iiint\limits_V F^2 dV$$

$$+2\left(M_9+2\sqrt{M_{10}}\right)^{-1}\Big\{\frac{3}{2m_2}\iiint\limits_V F^2 dV + \frac{1}{2}\iint\limits_{D(0)} G_{,j}a^{ij}G_{,i}\,ds$$

$$-\frac{1}{2}\iint\limits_S n^{-1}n_y\left(\frac{\partial\psi}{\partial t}\right)^2 ds + \frac{1}{\sqrt{M_{10}}}\iint\limits_S n_y^2\left[a_0 n(1-n_y^2)\right]^{-1}\left(\frac{\partial\psi}{\partial t}\right)^2 ds$$

$$\frac{1}{\sqrt{M_{10}}}\iint\limits_S (1-n_y^2)^{-1}\left(\frac{\partial\psi}{\partial T}\right)^2 ds - \frac{1}{2}\lim_{\substack{y\to y_0\\ y<y_0}}\iint\limits_{D(y)}\psi_{,j}a^{ij}\psi_{,i}\,ds\Big\}.$$

If the last term on the right of 3.110 is neglected, then the only unknown terms on the right of this inequality are the second and third. To estimate these unknown terms we make the transformation of variables given by 3.20 and observe that 3.29 is valid for our present problem. In place of 3.35 we write

3.111 $$\left(1 - \frac{M_3\alpha_2}{2a_0}\right)\iiint\limits_V a^{ij}\tilde{\varphi}_{,i}\tilde{\varphi}_{,j}\,dV$$

$$+\left(Km_2 - \frac{1}{2}M_4 - \tilde{M}_1 - \frac{1}{2\alpha_2} - \frac{\alpha_3}{2}\right)\iiint\limits_V \tilde{\varphi}^2 dV$$

$$\leq \frac{\alpha_4}{2}\iint\limits_S\left(\frac{\partial\psi}{\partial\nu}\right)^2 ds + \frac{1}{2\alpha_4}\iint\limits_S H^2 \exp\left[4K(y_0-y)\right]ds - \frac{1}{2}\iint\limits_{D(y_0)}\tilde{\varphi}^2 z\, ds$$

$$+\frac{1}{2}\iint_{D(0)} G^2 z \exp(2Ky_0)\,ds + \frac{1}{2\alpha_3}\iiint_V F^2 \exp[2K(y_0-y)]\,dV$$

$$-\frac{1}{2}\iint_S m_y z H^2 \exp[2K(y_0-y)]\,ds$$

In the last inequality, let

3.112 $$\alpha_2 = \frac{a_0}{M_3}$$

3.113 $$\alpha_3 = 1$$

3.114 $$K = \frac{1}{m_2}\Big\{\frac{M_4}{2} + \hat{M}_1 + \frac{M_3}{2a_0} + \frac{1}{2} + \frac{m_3}{4}\Big[C_1 + \frac{4M_7}{a_0 m_2 m_3}(M_9 + 2\sqrt{M_{10}})$$

$$+\frac{3M_1}{a_0} + \frac{M_3}{a_0} + \frac{m_3}{2}(M_9 + 2\sqrt{M_{10}})^{-1}\Big(\frac{3M_3}{2m_2 a_0} + \frac{M_8}{2a_0}\Big)\Big]^{-1}\Big[\frac{2\tilde{M}_1^2}{m_3} +$$

$$+\frac{3\hat{M}_1}{2m_3}(M_9 + 2\sqrt{M_{10}})^{-1}\Big]\Big\} = K_1$$

3.115 $$\alpha_4 = \frac{m_3}{8}\Big\{C_1 + \frac{4M_7}{a_0 m_2 m_3}(M_9 + 2\sqrt{M_{10}}) + \frac{3M_1}{a_0} + \frac{M_3}{a_0}$$

$$+\frac{m_3}{2}(M_9 + 2\sqrt{M_{10}})^{-1}\Big(\frac{3M_3}{2m_2 a_0} + \frac{M_8}{2a_0}\Big)\Big\}^{-1}$$

and then substitute the resulting inequality in 3.110. We note that the following terms appear on the right side of 3.110 after this substitution

3.116 $$-\frac{4\bar{m}_2}{m_3}\Big\{C_1 + \frac{4M_7}{a_0 m_2 m_3}\left(M_9 + 2\sqrt{M_{10}}\right) + \frac{3M_1}{a_0} + \frac{M_3}{a_0}$$

$$+ \frac{m_3}{2}\left(M_9 + 2\sqrt{M_{10}}\right)^{-1}\left(\frac{3M_3}{2m_2 a_0} + \frac{M_8}{2a_0}\right)\Big\}\iint_{D(y_0)} \psi^2 ds$$

$$-\lim_{\substack{y\to y_0\\ y<y_0}} \left(M_9 + 2\sqrt{M_{10}}\right)^{-1}\iint_{D(y)} a^{ij}\psi_{,i}\psi_{,j}\, ds$$

$$= -M_{11}\lim_{\substack{y\to y_0\\ y<y_0}} \iint_{D(y)} a^{ij}\psi_{,i}\psi_{,j}\, ds - M_{12}\iint_{D(y_0)} \psi^2 ds$$

where M_{11} and M_{12} are defined by the above inequality and

3.117 $$\bar{m}_2 = \min_{D(y_0)} \mathcal{Z}(x,y) > 0.$$

Since we are seeking an upper bound for the integral of the square of the conormal derivative over S, it is clear the unknown terms which appear in 3.116 may be neglected in the revised version of 3.110. In the original statement of the boundary value problem, the solution was assumed to be continuously differentiable in $V \cup S$; if we make the stronger assumption that the continuous differentiability holds in $V \cup S \cup \bar{D}(y_0)$, then it is easy to obtain an upper bound for 3.116 which is less than zero. In what follows the stronger assumption concerning differentiability is made since, if this condition does not apply, then one may clearly neglect 3.116 in order to obtain the desired result. We define

3.118 $$B(y_o) = \bar{D}(y_o) - D(y_o)$$

and then write

3.119 $$\int_{B(y_0)} f^i n_i \psi^2 dl = \iint_{D(y_0)} f^i_{,i} \psi^2 ds + 2\iint_{D(y_0)} f^i \psi \psi_{,i} ds$$

$$\leq (\bar{M}_2 + \alpha_5) \iint_{D(y_0)} \psi^2 ds + \frac{M_3}{\alpha_5 a_0} \iint_{D(y_0)} a^{ij} \psi_{,i} \psi_{,j} ds$$

where

3.120 $$\bar{M}_2 = \max_{D(y_o)} \left\{ f^i_{,i} \right\}.$$

Next we set

3.121 $$\frac{(\bar{M}_2 + \alpha_5) a_o \alpha_5}{M_3} = \frac{M_{12}}{M_{11}}$$

and solve for α_5 to obtain

3.122 $$\alpha_5 = \frac{1}{2}\left(-\bar{M}_2 + \sqrt{\bar{M}_2^2 + \frac{4 M_3 M_{12}}{a_0 M_{11}}}\right).$$

From the above it is clear that

3.123 $$-M_{11} \lim_{\substack{y \to y_0 \\ y < y_0}} \iint_{D(y)} a^{ij} \psi_{,i} \psi_{,j} ds - M_{12} \iint_{D(y_0)} \psi^2 ds$$

$$\leq -\frac{a_0 M_{11} \left(-\bar{M}_2 + \sqrt{\bar{M}_2^2 + \frac{4 M_3 M_{12}}{a_0 M_{11}}}\right)}{2 M_3} \int_{B(y_0)} f^i n_i H^2 dl < 0.$$

The inequality on the right of 3.123 is strict since in Chapter II the auxiliary functions f^i were chosen so that $f^i n_i$ had a positive minimum on S.

By combining the above, we are able to rewrite 3.110 as follows

3.124
$$\iint_S \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds \leq 4\mathcal{H} + \frac{8}{m_3} \iiint_V F^2 dv$$

$$+ 4\left(M_9 + 2\sqrt{M_{10}}\right)^{-1} \left\{ \frac{3}{2m_2} \iiint_V F^2 dv + \frac{1}{2} \iint_{D(0)} G_{,i} a^{ij} G_{,j} ds \right.$$

$$- \frac{1}{2} \iint_S n^{-1} n_y \left(\frac{\partial H}{\partial t}\right)^2 ds + \frac{1}{\sqrt{M_{10}}} \iint_S n_y^2 \{a_0 n (1-n_y^2)\}^{-1} \left(\frac{\partial H}{\partial t}\right)^2 ds$$

$$+ \frac{1}{\sqrt{M_{10}}} \iint_S (1-n_y^2)^{-1} \left(\frac{\partial H}{\partial T}\right)^2 ds + \frac{8}{m_3} \left\{ C_1 + \frac{4M_7}{a_0 m_2 m_3} (M_9 + 2\sqrt{M_{10}}) \right.$$

$$\left. + \frac{3M_1}{a_0} + \frac{M_3}{a_0} + \frac{m_3}{2} (M_9 + 2\sqrt{M_{10}})^{-1} \left(\frac{3M_3}{2m_2 a_0} + \frac{M_8}{2a_0} \right) \right\} \cdot$$

$$\cdot \left\{ \frac{1}{\alpha_4} \iint_S H^2 \exp[4K_1(y_0 - y)] ds + \iint_{D(0)} G^2 z \exp(2K_1 y_0) ds \right.$$

$$\left. + \iiint_V F^2 \exp[2K_1(y_0 - y)] dv - \iint_S n_y H^2 z \exp[2K_1(y_0 - y)] ds \right\}$$

$$- \frac{a_0 M_{11} \left(-\bar{M}_2 + \sqrt{\bar{M}_2^2 + \frac{4 M_3 M_{12}}{a_0 M_{11}}} \right)}{M_3} \int_{B(y_0)} f^i n_i H^2 dl = \mathcal{B}$$

where α_4 is given by 3.115 and the computable constant $\mathcal{B}$ is such that

3.125 $$\mathcal{B} = 0$$

when

3.126 $$F = H = G = 0.$$

Inequality 3.124 is a bound for the unknown factor of the last term on the right of 3.84. Bounds for the two remaining terms on the right of that inequality may now be obtained using the procedures of the first section of this chapter. With $\mathcal{B}$ known it is clear that 3.57 becomes

3.127
$$\lim_{\substack{y \to y_0 \\ y < y_0}} \iiint_{V(y)} \{ 2\mathcal{F} \tilde{\Psi} \bar{J}(\Psi) - \Psi^2 \bar{J}(\mathcal{F})$$

$$+ 2\mathcal{F} \tilde{\Psi}_{,i} a^{ij} \tilde{\Psi}_{,j} + 2kz\mathcal{F} \tilde{\Psi}^2 \} \, dV$$

$$\leq \iint_S \mathcal{F}^2 H^2 \exp[4k(y_0 - y)] \, ds + \mathcal{B}$$

$$- \iint_S \frac{\partial \mathcal{F}}{\partial \nu} H^2 \exp[2k(y_0 - y)] \, ds - \iint_S n_y \mathcal{F} z H^2 \exp[2k(y_0 - y)] \, ds$$

$$+ \iint_{D(0)} \mathcal{F} z G^2 \exp(2k y_0) \, ds = \mathcal{J}$$

The right side of 3.127 is computable and hence the desired bounds may be obtained in exactly the same manner as was employed in section 1 of this chapter. The results of these

where α_4 is given by 3.115 and the computable constant $\mathcal{B}$ is such that

3.125 $$\mathcal{B} = 0$$

when

3.126 $$F = H = G = 0.$$

Inequality 3.124 is a bound for the unknown factor of the last term on the right of 3.84. Bounds for the two remaining terms on the right of that inequality may now be obtained using the procedures of the first section of this chapter. With $\mathcal{B}$ known it is clear that 3.57 becomes

3.127 $$\lim_{\substack{y \to y_0 \\ y < y_0}} \iiint_{V(y)} \{ 2\mathcal{F}\tilde{\varphi}\,\bar{J}(\tilde{\varphi}) - \tilde{\varphi}^2\,\bar{J}(\mathcal{F}) + 2\mathcal{F}\tilde{\varphi}_{,i}\,a^{ij}\tilde{\varphi}_{,j} + 2kz\mathcal{F}\tilde{\varphi}^2 \}\, dV$$
$$\leq \iint_S \mathcal{F}^2 H^2 \exp[4k(y_0 - y)]\, ds + \mathcal{B} - \iint_S \frac{\partial \mathcal{F}}{\partial \nu} H^2 \exp[2k(y_0-y)]\, ds - \iint_S n_y \mathcal{F} z H^2 \exp[2k(y_0-y)]\, ds$$
$$+ \iint_{D(0)} \mathcal{F} z G^2 \exp(2ky_0)\, ds = \mathcal{J}$$

The right side of 3.127 is computable and hence the desired bounds may be obtained in exactly the same manner as was employed in section 1 of this chapter. The results of these

operations may be written as follows:

3.128 $$|W(x_0,y_0) - \chi(p)|$$

$$\leq \Big\{ \iiint_V [-\bar{J}(\mathcal{F})]^{-1} \bar{J}(\gamma_p)^2 dv \Big\}^{\frac{1}{2}} \Big\langle \Big\{ \iiint_V [\bar{J}(\mathcal{F})]^{-1} (\mathcal{F} F)^2 \cdot$$

$$\cdot \exp[2K_2(y_0 - y)]dv \Big\}^{\frac{1}{2}} + \Big\{ \mathcal{J}_2$$

$$+ \iiint_V [-\bar{J}(\mathcal{F})]^{-1} (\mathcal{F} F)^2 \exp[2K_2(y_0 - y)]dv \Big\}^{\frac{1}{2}} \Big\rangle$$

$$+ \Big\{ \iiint_V \gamma_p^2 \dot{\mathcal{F}}^{-1} \exp[-2K_3(y_0 - y)]dv \Big\}^{\frac{1}{2}} \Big\{ \tilde{\mathcal{J}} \Big\}^{\frac{1}{2}}$$

$$+ \Big\{ \iint_S \gamma_p^2 ds \Big\}^{\frac{1}{2}} \Big\{ \mathcal{B} \Big\}^{\frac{1}{2}}$$

where the functions γ_p and $\mathcal{F}$ are given by 2.10 and 2.39, the constants K_1, K_2, and K_3 by 3.114, 3.62 and 3.74, equation 3.127 defines $\mathcal{J}_i$ and the constant $\tilde{\mathcal{J}}$ is given by 3.75. The computable bound $\mathcal{B}$ is obtained from 3.124 while $\chi(p)$ is defined by 3.85. As in the problem of the previous section, we remark that if the function f is such that

3.129 $$f(x,y,w,\nabla w) = f(x,y)$$

then the result lends itself to improvement by means of the Rayleigh-Ritz technique.

In obtaining the conclusions of this section, the role of inequality 2.113 was basic. By referring to the derivation of that inequality it is clear that the symmetry

of the components of the matrix a^{ij} was essential; hence the consideration of a method for the symmetrization of an arbitrary problem is appropriate. The result is obtained by defining

3.130 $$\bar{a}^{ij} = \frac{1}{2}(a^{ij} + a^{ji})$$

and then, since

3.131 $$a^{ij}\psi_{,ij} = a^{ij}\psi_{,ji}$$

almost everywhere for $(x,y) \in V$, we may write

3.132 $$\begin{aligned}(a^{ij}\psi_{,i})_{,j} &= a^{ij}_{,j}\psi_{,i} + a^{ij}\psi_{,ij} \\ &= a^{ij}_{,j}\psi_{,i} + \tfrac{1}{2}(a^{ij} + a^{ji})\psi_{,ij} \\ &= a^{ij}_{,j}\psi_{,i} - \bar{a}^{ij}_{,j}\psi_{,i} + (\bar{a}^{ij}\psi_{,i})_{,j}\end{aligned}$$

where 3.132 is also valid almost everywhere in V. Due to the foregoing, the differential equation of the boundary value problem may be replaced almost everywhere by

3.133 $$\begin{aligned}(\bar{a}^{ij}\psi_{,i})_{,j} - z\frac{\partial\psi}{\partial y} &= f(x, y, \psi, \nabla\psi) + \bar{a}^{ij}_{,j}\psi_{,i} - a^{ij}_{,j}\psi_{,i} \\ &= \bar{f}(x, y, \psi, \nabla\psi).\end{aligned}$$

From 3.133 it is clear that $\bar{f}$ satisfies the same conditions as f and thus the original problem has been symmetrized as required.

Before concluding our consideration of normally parabolic problems, we remark that when the differential operator has the form

3.134 $$J(\psi) = (a^{ij}\,\psi_{,j})_{,i} - q\psi - z\frac{\partial \psi}{\partial y},$$

the transformation given by 3.20 yields

3.135 $$\tilde{J}(\tilde{\psi}) = (a^{ij}\,\tilde{\psi}_{,j})_{,i} - (q + K)\tilde{\psi} - z\frac{\partial \tilde{\psi}}{\partial y}.$$

Hence, if the minimum of the function q over V is a strictly positive number, then the constants K_1 of either this section or section one are reduced by the amount of this minimum. In order to appreciate the improvement which results under such conditions, the reader is referred to the example presented in Chapter VI.

CHAPTER IV

DEGENERATE PROBLEMS

1. The Dirichlet Problem at a Parabolic Point

The problems which have been considered thus far have been normally parabolic in character. We now turn our attention to problems which are normally parabolic at certain points and degenerate at others. As the above heading implies, this section pertains to the derivation of a bound for the solution of a degenerate boundary value problem at a normally parabolic point. We seek to bound the solution of

4.1
$$\begin{cases} J(w) = (a^{ij} w_{,i})_{,j} - qw - g\dfrac{\partial w}{\partial y} = f(x,y) & (x,y) \in V \\ W(x,o) = g(x) & (x,y) \in D(o) \\ W(x,y) = h(x,y) & (x,y) \in S \end{cases}$$

where w is continuously differentiable for $(x,y) \in V \cup S \cup D(y_o)$ and is such that its second derivative in x and mixed derivative in x and y are continuous in the interior of a finite number of subregions, the sum of which is D(y) for every y satisfying $o < y < y_o$. In addition it is necessary to assume that

4.2
$$\lim_{\substack{(x,y)\to(x,o)\\(x,y)\in V}} w(x,y) = g(x)$$

$$\lim_{\substack{(x,y)\to(x,o)\\(x,y)\in V}} w_{,i}(x,y) = g_i(x) \qquad i = 1, 2 \ldots N$$

The components of the symmetric matrix a^{ij} satisfy the conditions imposed in the second section of the previous chapter. The functions q and z are piecewise continuously differentiable in y for $(x,y) \in \overline{V}$ and z satisfies

4.3 $$z(x,y) \geq 0 \qquad (x,y) \in \overline{V}.$$

Prior to stating the additional conditions which are to be satisfied by the functions z and q, it is necessary to define a subregion E of $\overline{V}$ by

4.4 $$E = \left\{(x,y) \middle| z(x,y) = 0, \ (x,y) \in \overline{V}\right\}.$$

We suppose that q has the form

4.5 $$q = \varepsilon^I q_I = \sum_{I=1}^{\ell} \varepsilon^I q_I$$

where the constants ε^I, $i = 1, 2 \ldots \ell$, satisfy

4.6 $$\sum_{I=1}^{\ell} \varepsilon^I = 1$$

and the functions q_I, $i = 1, 2 \ldots \ell$, fulfill certain conditions which will be prescribed in what follows. Let the coordinates $\underline{x}^i$, $i = 1, 2 \ldots N$, be chosen to satisfy

4.7 $$\min_{(\underline{x},y)} \ \max_{(x,y)\in S} \sum_{1}^{N} (x^i - \underline{x}^i)^2 = \underline{R}^2$$

where $o \leq y \leq y_o$. For N = 1 we suppose that

4.8 a $$\tilde{m} = \min_{E_\delta}\left\{o; q > -a_o (4\underline{R}^2)^{-1}\right\}$$

with $E_\delta \subset V$ defined by

4.9 $\quad E_\delta = \left\{ (x,y) \,\middle|\, \|(x,y) - \bar{p}\| < \delta \text{ for some } \bar{p} \in E \text{ with } \delta > 0 \right\} \cap V$

In 4.9 the symbol $\| \ \|$ denotes the Euclidean distance in the 2 (and later N + 1) dimensional space of the problem. If on the other hand N = 2, then q must satisfy

4.8 b $\quad \tilde{m}_I = \min_{E_\delta} \left\{ 0; r_I q_I > -a_0 (4R_I)^{-1} \right\} \quad I = 1, 2 \ldots \ell$

where

4.10 $\quad r_I = \left\{ \sum_{i=1}^{N} (x^i - x_I^i)^2 \right\}^{\frac{1}{2}}$

and

4.11 $\quad R_I = \max_{\bar{V}} r_I \,.$

Finally for $N \geq 3$

4.8 c $\quad \tilde{m}_I = \min_{E_\delta} \left\{ 0; r_I^2 q_I > -\frac{1}{4} a_0 (N-2)^2 \right\} \quad I = 1, 2 \ldots \ell$

Since the techniques which are used to obtain solution bounds for N = 1 or N = 2 are very similar to those which are employed for $N \geq 3$, the method shall be presented in complete detail only for the latter case. Thus for $N \geq 3$ we define $\bar{m}$ by

4.12 $\quad 0 \leq -4 \, \Sigma^I \tilde{m}_I \left[a_0 (N-2)^2 \right]^{-1} = \bar{m} < 1$

If N = 1 or N = 2, similar constants are introduced.

Previously we required that Z be piecewise continuously differentiable in y for $(x,y) \in \bar{V}$; in addition to this,

it is also necessary that $\frac{\partial z}{\partial y}$ be continuous in $\bar{V}$ for every $(x,y) \in \dot{E}$ where $\dot{E}$ is the boundary of E, and $\frac{\partial z}{\partial y} = 0$ on $E \cap (\bar{V} - V)$.

In 4.1 the boundary function f = f(x,y) is continuous in y for $(x,y) \in \bar{V}$ and the first derivative of f with respect to y is continuous in the interior of a finite number of subintervals, the sum of which is the line $(\bar{x}, y)$ for every fixed $\bar{x}$ where $(\bar{x}, y) \in V$. In addition we suppose that $\frac{\partial f}{\partial y}$ is integrable and square integrable over V and that g and $g_{,1}$ are integrable and square integrable over D(o). By definition it is clear that f is integrable and square integrable over D(o), $D(y_o)$, S and V.

It is now our aim to determine a bound for the solution of 4.1. As the heading of this section indicates, we assume that

4.13 $$z(x,y) > 0 \qquad (x,y) \in D(y_o)$$

and that 2.9 is satisfied at the point $p \in D(y_o)$ at which a bound is desired. The arbitrary function $\varphi = \varphi(x,y)$ is chosen as in the previous chapter and the function $\psi(x,y)$ is introduced where

4.14 $$\psi(x,y) = w(x,y) - \varphi(x,y)$$

We compute

4.15 $$\begin{cases} J(\psi) = F(x,y) & (x,y) \in V \\ \psi(x,o) = G(x) & (x,y) \in D(o) \\ \psi(x,y) = H(x,y) & (x,y) \in S \end{cases}$$

Let

4.16 $$\bar{J}(\psi) = (a^{ij}\psi_{,i})_{,j} - q\psi + \frac{\partial}{\partial y}(z\psi),$$

and then with γ_p given by 2.10 we note that 3.15 is valid for the present problem. From this it follows that

4.17 $$|W(x_0, y_0) - \chi(p)|$$

$$\leq \left| \lim_{y \to y_0} \iiint_{V(y)} \psi \, \bar{J}(\gamma_p) \, dv \right|$$

$$+ \left\{ \iint_S \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds \right\}^{1/2} \left\{ \iint_S \gamma_p^2 \, ds \right\}^{1/2}$$

where

4.18 $$\chi(p) = \varphi(x_0, y_0) - \iint_S H \frac{\partial \gamma_p}{\partial \nu} ds$$

$$- \lim_{y \to y_0} \iiint_{V(y)} \gamma_p F \, dv + \iint_{D(0)} \gamma_p z G \, ds - \iint_S n_y \gamma_p z H \, ds .$$

Since F is bounded over $\bar{V}$, it is clear that the volume integral in 4.18 exists; thus the desired bound is available when the terms on the right of 4.17 are estimated. We shall first consider the integral of the square of the conormal derivative over S and then use this bound to estimate the unknown volume integral. For this purpose we make the transformation of variables given by 3.20. The boundary value problem then becomes

$$\tilde{J}(\tilde{\varphi}) = (a^{ij}\tilde{\varphi}_{,i})_{,j} - \bar{q}\,\tilde{\varphi} - z\frac{\partial\tilde{\varphi}}{\partial y} = F\exp[K(y_0-y)] \qquad (x,y)\in V$$

4.19
$$\tilde{\varphi}(x,0) = G\exp(Ky_0) \qquad (x,y)\in D(0)$$

$$\tilde{\varphi}(x,y) = H\exp[K(y_0-y)] \qquad (x,y)\in S$$

where

4.20
$$\bar{q} = KZ + q_0 .$$

Due to the fact that $\frac{\partial Z}{\partial y}$ is continuous in $\bar{V}$ for $(x,y)\in \dot{E}$ and 4.8 is satisfied, we may choose $K > 0$ so that

4.21 c
$$\tilde{m}_I = \min_V \left\{0\,;\, r_I^2\left(\bar{q}_I - KZ[2\ell\varepsilon^I]^{-1}\right) > -\tfrac{1}{4}a_0(N-2)^2\right\} \qquad I = 1,2,\dots\ell$$

for $N \geqslant 3$ and

4.22
$$\min_V \left(\tfrac{1}{2}KZ - \tfrac{1}{2}\frac{\partial Z}{\partial y}\right) > -(1-\bar{m})\,\frac{a_0 N^2}{8\,\underline{R}^2}$$

where

4.23
$$\bar{q} = KZ + \varepsilon^I q_I = \sum_{I=1}^{\ell}\varepsilon^I\left(\frac{KZ}{\ell\varepsilon^I} + q_I\right) = \varepsilon^I\bar{q}_I$$

In the above we have made use of the fact that $\frac{\partial Z}{\partial y} = 0$ on $\dot{E}$. Similarly the constant $K > 0$ is chosen so that

4.21 a
$$\tilde{m} = \min_V \left\{\bar{q} - \tfrac{1}{2}ZK > -a_0(4\underline{R}^2)^{-1}; 0\right\} \quad \text{for } N = 1$$

and

4.21 b $$\tilde{m}_I = \min_V \left\{ 0 \; ; \; r_I \left(\bar{q}_I - Kz[2d\varepsilon^I]^{-1} \right) > -a_0 (4R_I)^{-1} \right\}$$

$I = 1, 2, \ldots d$ and $N = 2$

With $K > 0$ as chosen above we have from 2.113

4.24 $$\iint_S \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds \leq \iint_S \left(\frac{\partial \tilde{\psi}}{\partial \nu}\right)^2 ds$$

$$\leq \mathcal{H} + \frac{2}{m_3}\left(C_1 + \frac{2M_1}{a_0} + \frac{M_{13}}{\alpha_1 a_0}\right) \iiint_V a^{ij} \tilde{\psi}_{,i} \tilde{\psi}_{,j} \, dV$$

$$+ \frac{2M_{14}}{m_3} \iiint_V \tilde{\psi}^2 dV + \frac{2}{m_3} \iiint_V F^2 \exp[2K(y_0 - y)] \, dV$$

$$+ \frac{2\alpha_1}{m_3} \iiint_V z \left(\frac{\partial \tilde{\psi}}{\partial y}\right)^2 dV$$

where α_1 is an arbitrary positive number and

4.25 $$M_{13} = \max_V \left\{ z \sum_1^N (f^i)^2 \right\}$$

4.26 $$M_{14} = \max_V \left\{ \bar{q}^2 \right\}$$

In order to estimate the unknown terms on the right of 4.24 we observe that for any set of N functions g^i, $i = 1, 2 \ldots N$, which are piecewise continuously differentiable in x for $(x, y) \in \bar{V}$, it follows that

4.27 $$\iint_S \{ g^i n_i \tilde{\psi}^2 \} ds = \iiint_V g^i_{,i} \tilde{\psi}^2 dV + 2 \iiint_V g^i \psi_{,i} \psi \, dV.$$

By setting

4.28 $$g^i = (x^i - \underline{x}^i)$$

in 4.27, it is easily shown that

4.29 $$\iiint_V \tilde{\Psi}^2 dv \leq \frac{2}{N} \iint_S (x^i - \underline{x}^i) n_i \tilde{\Psi}^2 ds + \frac{4R^2}{N^2 a_0} \iiint_V a^{ij} \tilde{\Psi}_{,i} \tilde{\Psi}_{,j} dv.$$

Substitution of this inequality into 4.24 yields

4.30 $$\iint_V \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds$$

$$\leq \mathcal{H} + \frac{2}{m_3}\left(C_1 + \frac{2M_1}{a_0} + \frac{M_{13}}{\alpha_1 a_0} + \frac{4M_{14}R^2}{N^2 a_0}\right) \iiint_V a^{ij} \tilde{\Psi}_{,i} \tilde{\Psi}_{,j} dv$$

$$+ \frac{4M_{14}}{m_3 N} \iint_S (x^i - \underline{x}^i) n_i H^2 \exp[2K(y_0 - y)] ds$$

$$+ \frac{2}{m_3} \iiint_V F^2 \exp[2K(y_0 - y)] dV + \frac{2\alpha_1}{m_3} \iiint_V z \left(\frac{\partial \tilde{\Psi}}{\partial y}\right)^2 dv.$$

To bound the unknown terms on the right of 4.30 we write

4.31 $$\iiint_V \frac{\partial \tilde{\Psi}}{\partial y} \tilde{F}(\tilde{\Psi}) dV = \iint_{D(y_0)} \tilde{\Psi} F ds - \iint_{D(0)} GF \exp(2Ky_0) ds$$

$$- \iiint_V \tilde{\Psi} \frac{\partial F}{\partial y} \exp[K(y_0 - y)] dV + \iiint_V K \tilde{\Psi} F \exp[K(y_0 - y)] dV$$

$$+ \iint_S n_y HF \exp[2K(y_0 - y)] ds = - \iiint_V \frac{\partial \tilde{\Psi}_{,j}}{\partial y} a^{ij} \tilde{\Psi}_{,i} dV$$

$$+ \iint_S \frac{\partial \tilde{\Psi}}{\partial y} \frac{\partial \tilde{\Psi}}{\partial \nu} ds - \frac{1}{2} \iint_{D(y_0)} \bar{q} \tilde{\Psi}^2 ds - \frac{1}{2} \iint_S n_y \bar{q} H^2 \exp[2K(y_0 - y)] ds$$

$$+\frac{1}{2}\iint_{D(0)} \bar{q}\, G^2 \exp(2Ky_0)\, ds + \frac{1}{2}\iiint_V \tilde{\psi}^2 \frac{\partial \bar{q}}{\partial y}\, dv - \iiint_V z\left(\frac{\partial \tilde{\psi}}{\partial y}\right)^2 dv.$$

Due to the assumptions which were made concerning the differentiability of the solution of 4.1, it is clear that 3.93, 3.95, 3.97, and 3.105 are valid for the problem presently under investigation. Substitution of these results into a transposed form of 4.31 gives after making use of the fact that $\psi = \tilde{\psi}$ on $D(y_0)$

4.32
$$\iiint_V z\left(\frac{\partial \tilde{\psi}}{\partial y}\right)^2 dv + \frac{1}{2}\left\{\iint_{D(y_0)} \psi_{,i}\, a^{ij} \psi_{,j}\, ds + \iint_{D(y_0)} (\bar{q} - \varepsilon_1)\, \psi^2\, ds\right\}$$

$$\leq \frac{1}{2}\left(M_9 + 2\sqrt{M_{10}}\right) \exp(2Ky_0) \iint_S \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds + \frac{1}{2\varepsilon_1} \iint_{D(y_0)} F^2\, ds$$

$$+\left(\frac{M_8}{2a_0} + \frac{4R^2}{N^2 a_0} + \frac{2R^2 M_{15}}{N^2 a_0}\right) \iiint_V a^{ij} \tilde{\psi}_{,i} \tilde{\psi}_{,j}\, dv + \mathfrak{M}_1$$

where we assume that $(M_9 + 2\sqrt{M_{10}})$ is a strictly positive number. If this is not the case, obvious modifications in what follows will yield the same result. In 4.32 it is clear that

4.33 $$\mathfrak{M}_1 = -\iint_S n_y H F \exp 2K(y_0 - y)\, ds + \iint_{D(0)} G F \exp(2Ky_0)\, ds$$

$$+ \tfrac{1}{2} \iiint_V \left[\frac{\partial F}{\partial y} - KF\right]^2 \exp[2K(y_0 - y)]\, dV$$

$$+ \tfrac{1}{2} \iint_{D(0)} \bar{q}\, G^2 \exp(2Ky_0)\, ds - \tfrac{1}{2} \iint_S n_y \bar{q}\, H^2 \exp[2K(y_0 - y)]\, ds$$

$$+ \tfrac{1}{2} \iint_{D(0)} G_{,i} a^{ij} G_{,j} \exp(2Ky_0)\, ds + \tfrac{1}{N}(2 + M_{15}) \iint_S (x^i - \underline{x}^i) n_i H^2 \exp[2K(y_0 - y)]\, ds$$

$$- \tfrac{1}{2} \iint_S m^{-1} n_y \left(\frac{\partial H}{\partial t}\right)^2 \exp[2K(y_0 - y)]\, ds$$

$$+ (M_{10})^{-\frac{1}{2}} \iint_S [1 - n_y^2]^{-1} \left(\frac{\partial}{\partial T} H \exp[K(y_0 - y)]\right)^2 ds$$

$$+ (M_{10})^{-\frac{1}{2}} \iint_S n_y^2 [q_0 m (1 - n_y^2)]^{-1} \left(\frac{\partial H}{\partial t}\right)^2 \exp[2K(y_0 - y)]\, ds$$

and

4.34 $$M_{15} = \max_V \left\{ \frac{\partial \bar{q}}{\partial y} \right\}.$$

We now show that due to the form of q and hence $\bar{q}$ it is possible to choose ε_1 so that the terms in the braces on the left of 4.32 may be bounded from below by a computable constant. On the surface $D(y_0)$ we have by means of the divergence theorem

4.35 $$\int_{B(y_0)} g^i n_i \psi^2 \, dl$$

$$= \iint_{D(y_0)} g^i_{,i} \psi^2 \, ds + 2 \iint_{D(y_0)} g^i \psi_{,i} \psi \, ds$$

where $B(y_0)$ is the boundary of $D(y_0)$. For $N \geq 3$ the form of 4.35 leads us to define

4.36 $$g^i = (x^i - x^i_I) r_I^{-2}$$

and then for every I obtain from 4.35

4.37 $$\iint_{D(y_0)} \psi^2 r_I^{-2} \, ds$$

$$\leq \frac{\alpha}{(N-2)\alpha - 1} \int_{B(y_0)} (x^i - x^i_I) r_I^{-2} n_i \psi^2 \, dl + \frac{\alpha^2}{a_0[(N-2)\alpha - 1]} \iint_{D(y_0)} a^{ij} \psi_{,i} \psi_{,j} \, ds$$

where α is an arbitrary positive number satisfying

4.38 $$\alpha > \frac{1}{N-2}$$

We choose α so that the coefficient of the last term on the right of 4.37 is minimized. The above equation thus becomes

4.38 c
$$\iint_{D(y_0)} \psi^2 r_I^{-2}\, ds$$

$$\le \frac{2}{N-2} \int_{B(y_0)} (x^i - x_I^i) r_I^{-2} n_i \psi^2\, dl + \frac{4}{(N-2)^2 a_0} \iint_{D(y_0)} a^{ij} \psi_{,i} \psi_{,j}\, ds .$$

For later use we observe that it is also possible to show that

4.39 c
$$\iiint_V \tilde{\psi}^2 r_I^{-2}\, dV$$

$$\le \frac{2}{N-2} \iint_S (x^i - x_I^i) r_I^{-2} n_i \tilde{\psi}^2\, ds + \frac{4}{(N-2)^2 a_0} \iiint_V a^{ij} \tilde{\psi}_{,i} \tilde{\psi}_{,j}\, dV$$

For N = 1 we have

4.38 a
$$\int_{D(y_0)} \psi^2 ds \le 2\,(x - \underline{x})\,\psi^2 \Big]_{x=x_1}^{x=x_2} + \frac{4\underline{R}^2}{a_0} \int_{D(y_0)} a^{11} \psi_{,x} \psi_{,x}\, ds$$

and

4.39 a
$$\iint_V \tilde{\psi}^2 dV \le 2 \int_S (x - \underline{x}) n_x \tilde{\psi}^2 ds + \frac{4\underline{R}^2}{a_0} \iint_V a^{11} \psi_{,x} \psi_{,x}\, dV$$

Finally for N = 2 we may write

4.38 b
$$\iint_{D(y_0)} \psi^2 r_I^{-1}\, ds \le 2 \int_{B(y_0)} (x^i - x_I^i) r_I^{-1} n_i \psi^2\, dl + \frac{4R_I}{a_0} \iint_{D(y_0)} a^{ij} \psi_{,i} \psi_{,j}\, ds$$

and

4.39 b
$$\iiint_V \tilde{\psi}^2 r_I^{-1}\, dV \le 2 \iint_S (x^i - x_I^i) r_I^{-1} n_i \tilde{\psi}^2\, ds + \frac{4R_I}{a_0} \iiint_V a^{ij} \tilde{\psi}_{,i} \tilde{\psi}_{,j}\, dV.$$

We are now in a position to obtain a lower bound for the terms in the braces on the left of 4.32. The details shall be worked out only for $N \geq 3$. Similar results easily follow in the other two cases. By means of the above and 4.21 c

4.40
$$\iint_{D(y_0)} \bar{q}\, \psi^2 ds \geq \iint_{D(y_0)} \sum_{I=1}^{\ell} \varepsilon^I \tilde{m}_I \psi^2 r_I^{-2} ds + \frac{1}{2} \iint_{D(y_0)} K z \psi^2 ds$$

$$\geq \sum_{I=1}^{\ell} \varepsilon^I \tilde{m}_I \left(\frac{2}{N-2}\right) \int_B (x^i - x_I^i) r_I^{-2} n_i H^2 d\ell - \bar{m} \iint_{D(y_0)} a^{ij} \psi_{,i} \psi_{,j} ds$$

We notice that the line integral on the right of 4.40 is actually a volume integral over an (N-1) dimensional domain; and, since $N \geq 3$, it is clear that the integral exists. In 4.32 we choose

4.41
$$\varepsilon_1 = (1-\bar{m}) \frac{N^2 a_0}{4\underline{R}^2} > 0$$

where the strict inequality on the right follows from 4.12. Substitution of 4.41 into 4.32 produces the inequality

4.42
$$\iiint_V z \left(\frac{\partial \tilde{\psi}}{\partial y}\right)^2 dV \leq \frac{1}{2}\left(M_9 + 2\sqrt{M_{10}}\right) \exp(2Ky_0) \iint_S \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds$$

$$+ 2\underline{R}^2 \left[N^2 a_0 (1-\bar{m})\right]^{-1} \iint_{D(y_0)} F^2 ds + \left(\frac{M_8}{2a_0} + \frac{4\underline{R}^2}{N^2 a_0} + \frac{2\underline{R}^2 M_{15}}{N^2 a_0}\right) \iiint_V a^{ij} \tilde{\psi}_{,i} \tilde{\psi}_{,j} dV$$

$$-\sum_{I=1}^{\ell} \varepsilon^I \tilde{m}_I \left(\frac{2}{N-2}\right) \int_{B(y_0)} (x^i - x^i_I) r_I^{-2} n_i H^2 \, d\ell$$

$$+ (1-\bar{m}) \frac{Na_0}{4\underline{R}^2} \int_{B(y_0)} (x^i - \underline{x}^i) n_i H^2 \, d\ell \quad + \mathcal{M}_1$$

In deriving the above, an inequality analogous to 4.29 over $D(y_o)$ is used to bound the integral of the square of ψ over $D(y_o)$.

So that 4.30 may yield a non-trivial bound for the integral of the square of the conormal derivative, we choose α_1 as follows:

4.43 $$\alpha_1 = \tfrac{1}{2} m_3 (M_9 + 2\sqrt{M_{10}})^{-1} \exp(-2Ky_o)$$

Inequality 4.30 then gives

4.44 $$\iint_S \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds \le \frac{4}{m_3}\Big\{C_1 + \frac{2M_1}{a_0} + \frac{2M_{13}}{a_0 m_3}(M_9 + 2\sqrt{M_{10}}) \exp(2Ky_0)$$

$$+ \frac{4M_{14}\underline{R}^2}{N^2 a_0} + \frac{1}{2} m_3 \left(\frac{M_8}{2a_0} + \frac{4\underline{R}^2}{N^2 a_0} + \frac{2\underline{R}^2 M_{15}}{N^2 a_0}\right)(M_9 + 2\sqrt{M_{10}})^{-1} \cdot$$

$$\cdot \exp(-2Ky_0)\Big\} \iiint_V a^{ij} \tilde{\varphi}_{,i} \tilde{\varphi}_{,j} \, dv + \mathcal{M}_2$$

$$= \mathcal{M}_3 \iiint_V a^{ij} \tilde{\varphi}_{,i} \tilde{\varphi}_{,j} \, dv + \mathcal{M}_2$$

where

4.45 $$\mathcal{M}_2 = 2\mathcal{K} + \frac{8M_{14}}{m_3 N}\iint_S (x^i - \underline{x}^i) n_i H^2 \exp[2K(y_0 - y)]\,ds$$

$$+\frac{4}{m_3}\iiint_V F^2 \exp[2K(y_0-y)]\,dv + 2(M_9 + 2\sqrt{M_{10}})^{-1}\exp(-2Ky_0)\{2\underline{R}^2[N^2 a_0(1-\bar{m})]^{-1}$$

$$\cdot\iint F^2 ds + \mathcal{M}_1 - \sum_{I=1}^{\ell}\varepsilon^I \tilde{m}_I\left(\frac{2}{N-2}\right)\int_{B(y_0)}(x^i - x^i_I) r_I^{-2} n_i H^2\, d\ell$$

$$+(1-\bar{m})\frac{N a_0}{4\underline{R}^2}\int_{B(y_0)}(x^i - \underline{x}^i) n_i H^2\, d\ell\}$$

The unknown term appearing on the right of 4.44 is estimated by using the divergence theorem to write

4.46 $$\iiint_V a^{ij}\tilde{\varphi}_{,i}\tilde{\varphi}_{,j}\,dv = -\iiint_V \tilde{\varphi}\,\tilde{J}(\tilde{\varphi})\,dv$$

$$+\iint_S \tilde{\varphi}\frac{\partial\tilde{\varphi}}{\partial\nu}ds - \iiint_V \bar{q}\,\tilde{\varphi}^2 dv - \frac{1}{2}\iint_{D(y_0)} z\tilde{\varphi}^2 ds$$

$$-\frac{1}{2}\iint_S n_y z\tilde{\varphi}^2 ds + \frac{1}{2}\iint_{D(0)} z\tilde{\varphi}^2 ds + \frac{1}{2}\iiint_V \frac{\partial z}{\partial y}\tilde{\varphi}^2 dv$$

$$\le + \frac{2\varepsilon_1 \underline{R}^2}{N^2 a_0}\iiint_V a^{ij}\tilde{\varphi}_{,i}\tilde{\varphi}_{,j}\,dv + \frac{\varepsilon_1}{N}\iint_S (x^i - \underline{x}^i) n_i H^2 \exp[2K(y_0-y)]\,ds$$

$$+\frac{1}{2\varepsilon_1}\iiint_V F^2\exp[2K(y_0-y)]\,dv + \frac{\varepsilon_2}{2}\iint_S\left(\frac{\partial\psi}{\partial\nu}\right)^2 ds$$

$$+\frac{1}{2\varepsilon_2}\iint_S H^2\exp[4K(y_0-y)]\,ds$$

$$-\sum_{I=1}^{\ell}\varepsilon^I\tilde{m}_I\left(\frac{2}{N-2}\right)\iint_S (x^i - x^i_I) r_I^{-2} n_i H^2\exp[2K(y_0-y)]\,ds$$

$$+\bar{m}\iiint_V a^{ij}\tilde{\varphi}_{,i}\tilde{\varphi}_{,j}\,dv + \frac{1}{2}\iint_{D(0)} z G^2\exp(2Ky_0)\,ds$$

$$-\frac{1}{2}\iint_S n_y z H^2 \exp[2K(y_0-y)]\,ds + \frac{1}{2}(1-\bar{m})\iiint_V a^{ij}\tilde{\varphi}_{,i}\tilde{\varphi}_{,j}\,dV$$

$$+(1-\bar{m})\frac{a_0 N}{4\underline{R}^2}\iint_S (x^i-\underline{x}^i)\,n_i H^2 \exp[2K(y_0-y)]\,ds$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are arbitrary positive numbers which shall be suitably chosen in what follows. To obtain the above it is clear that 4.21 c, 4.22, 4.29 and 4.38 c have been used as necessary. Inequality 4.46 is now transposed so that the unknown term on the right of 4.44 may be bounded

4.47

$$\frac{1}{4}(1-\bar{m})\iiint_V a^{ij}\tilde{\varphi}_{,i}\tilde{\varphi}_{,j}\,dV$$

$$\leq \frac{\mathcal{E}_2}{2}\iint_S \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds + \frac{1}{2\mathcal{E}_2}\iint_S H^2 \exp[4K(y_0-y)]\,ds$$

$$+\left(\frac{1-\bar{m}}{8}\right)\frac{Na_0}{\underline{R}^2}\iint_S (x^i-\underline{x}^i)\,n_i H^2 \exp[2K(y_0-y)]\,ds$$

$$+\left(\frac{4}{1-\bar{m}}\right)\frac{\underline{R}^2}{N^2 a_0}\iiint_V F^2 \exp[2K(y_0-y)]\,dV$$

$$-\sum_{I-1}^{\ell}\varepsilon^I \tilde{m}_I\left(\frac{2}{N-2}\right)\iint_S (x^i-x^i_I)\,r_I^{-2} n_i H^2 \exp[2K(y_0-y)]\,ds$$

$$+\frac{1}{2}\iint_{D(0)} z G^2 \exp(2Ky_0)\,ds - \frac{1}{2}\iint_S n_y z H^2 \exp[2K(y_0-y)]\,ds$$

$$+(1-\bar{m})\frac{a_0 N}{4\underline{R}^2}\iint_S (x^i-\underline{x}^i)\,n_i H^2 \exp[2K(y_0-y)]\,ds$$

$$= \frac{\mathcal{E}_2}{2}\iint_S \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds + \frac{1}{2\mathcal{E}_2}\iint_S H^2 \exp[4K(y_0-y)]\,ds + \mathcal{M}_4$$

where $\mathcal{M}_4$ is defined by the above equation and it is clear that we have set

4.48 $$\mathcal{E}_1 = \left(\frac{1-\overline{m}}{8}\right) \frac{N^2 a_0}{\underline{R}^2}$$

Since $\overline{m} < 1$ due to 4.12, inequality 4.47 provides a non-trivial bound for the integral of $a^{ij}\,\tilde{\Psi}_{,i}\,\tilde{\Psi}_{,j}$. By comparing 4.44 and 4.47 we are led to choose

4.49 $$\mathcal{E}_2 = \left(\frac{1-\overline{m}}{8}\right)\mathcal{M}_3^{-1}$$

then 4.44 becomes

4.50 $$\iint_S \left(\frac{\partial \psi}{\partial \nu}\right)^2 ds \leq 2\mathcal{M}_2 + \mathcal{M}_3^2\left(\frac{4}{1-\overline{m}}\right)^2 \iint_S H^2 \exp[4K(y_0-y)]\,ds$$

$$+ \left(\frac{8}{1-\overline{m}}\right)\mathcal{M}_3\mathcal{M}_4 = \mathcal{B}$$

where $\mathcal{B}$ is computable and is such that

4.51 $$\mathcal{B} = 0$$

if

4.52 $$F = H = G = 0$$

The above is a bound for the unknown factor of the second term on the right of 4.17. We next bound the volume integral on the right of that inequality in terms of computable constants and 4.50. Let

4.53 $$J^{*}(\psi) = (a^{ij}\,\psi_{,i})_{,j} - 2q\psi + \frac{\partial}{\partial y}(s\psi)$$

and consider the function $\mathcal{F}$ given by 2.74. Since the lower order terms of J^* and $\bar{J}$ are identical, we may deduce from Theorem 2.3 that the arbitrary constants of 2.74 may be chosen so that

4.54 $$-\mathcal{F} \leq M \quad \text{and} \quad J^*(\mathcal{F}) \leq 0 \qquad \text{for } (x,y) \in \overline{V}$$

and

4.55 $$J^*(\mathcal{F}) < 0$$

for every $(x,y) \in V$ such that $0 < \varepsilon \leq y_0 - y \leq y_0$. From the proof of Theorem 2.1 and the definition of ψ and z, it follows that

4.56 $$\lim_{\substack{y \to y_0 \\ y < y_0}} \iint_{D(y)} \mathcal{F} z \psi^2 ds = 0$$

By means of the divergence theorem and the differentiability of $\mathcal{F}$ we have

4.57 $$\lim_{\substack{y \to y_0 \\ y < y_0}} \iiint_{V(y)} \{\mathcal{F} J(\psi^2) - \psi^2 J^*(\mathcal{F})\} dv - \iiint_V q \psi^2 \mathcal{F} dv$$

$$= \iint_S \{\mathcal{F} \frac{\partial \psi^2}{\partial \nu} - \psi^2 \frac{\partial \mathcal{F}}{\partial \nu}\} ds - \iint_S n_y \mathcal{F} z \psi^2 ds$$

$$+ \iint_{D(0)} \mathcal{F} z \psi^2 ds$$

and by expansion of the last equation the following inequality is established

4.58 $$\lim_{\substack{y\to y_0\\ y<y_0}} \iiint_{V(y)} \{2\mathcal{F}\psi J(\psi) - \psi^2 J^*(\mathcal{F})\}\, dv \leq -2\iiint_V \mathcal{F} a^{ij}\psi_{x_i}\psi_{x_j}\, dv$$

$$+\iint_S \{\mathcal{F}^2 - \frac{\partial \mathcal{F}}{\partial \nu} - m_y z\mathcal{F}\} H^2 ds + \iint_{D(0)} \mathcal{F} z G^2 ds + \mathcal{B}$$

$$\leq 2M\mathcal{D} + \iint_S \{\mathcal{F}^2 - \frac{\partial \mathcal{F}}{\partial \nu} - m_y z\mathcal{F}\} H^2 ds + \iint_{D(0)} \mathcal{F} z G^2 ds + \mathcal{B} = \mathcal{J}$$

where $\mathcal{D}$ is obtained from 4.47 after application of 4.50 and M is given by 4.54. By an argument similar to that which was used to establish 3.64, we are able to write

4.59 $$2\iiint_V \mathcal{F}\psi J(\psi)\, dv - \iiint_V \psi^2 J^*(\mathcal{F})\, dv \leq \mathcal{J}$$

and finally from equations analogous to 3.66, 3.67, 3.68 and 3.69 we have

4.60 $$\left|\lim_{y\to y_0} \iiint_{V(y)} \psi \bar{J}(\gamma_p)\, dv\right|$$

$$\leq \left\{\iiint_V [-J^*(\mathcal{F})]^{-1} \bar{J}(\gamma_p)^2 dv\right\}^{1/2} \left\langle \left\{\iiint_V [-J^*(\mathcal{F})]^{-1}\mathcal{F}^2 F^2 dv\right\}^{1/2}\right.$$

$$\left. + \left\{\mathcal{J} + \iiint_V [-J^*(\mathcal{F})]^{-1}\mathcal{F}^2 F^2 dv\right\}^{1/2}\right\rangle.$$

The existence of the integrals on the right of 4.60 is established by a discussion similar to that of the previous chapter.

Inequalities 4.51 and 4.60 give computable bounds for the unknown terms on the right of 4.17 and hence the desired pointwise bound is obtained from the inequality

4.61 $$|w(x_0,y_0)-\chi(p)| \leq \left\{\iiint_V [-J^*(\mathcal{F})]^{-1} \bar{J}(\gamma_p)^2 dv\right\}^{1/2} \cdot \left\langle \left\{\iiint_V [-J^*(\mathcal{F})]^{-1} \mathcal{F}^2 F^2 dv\right\}^{1/2} + \left\{\mathcal{J} + \iiint_V [-J^*(\mathcal{F})]^{-1} \mathcal{F}^2 F^2 dv\right\}^{1/2} \right\rangle + \left\{\iint_S \gamma_p^2 ds\right\}^{1/2} \left\{\mathcal{B}\right\}^{1/2}$$

where the functions γ_p and $\mathcal{F}$ are given by 2.10 and 2.74 and the computable constants $\mathcal{B}$ and $\mathcal{J}$ are obtained from 4.50 and 4.58. As in the previous problems, we observe that the right side of 4.61 is identically equal to zero if $F = G = H = 0$.

During the investigation of the problem of this section, an attempt was made to generalize the method by omitting the requirement that $\frac{\partial z}{\partial y}$ be continuous in V for $(x,y) \in \dot{E}$. Such a generalization has not as yet been achieved; however, if the domain V is restricted as indicated below, then the desired bound is available.

We suppose that the given boundary value problem is identical to that stated at the beginning of this except that $\frac{\partial z}{\partial y}$ need not be continuous in V for $(x,y) \in \dot{E}$. In addition, however, we suppose that V is a cylinder (that is, $D(o) = D(y)$ for $o \leq y \leq y_o$). It is evident that equations

4.14 to 4.20 are valid. We choose $K > o$ so that

4.62 $$\tilde{m}_I = \min_V \left\{ o, r_I^2 \bar{q}_I > -\frac{1}{4} a_o (N-2)^2 \right\} \quad I = 1, 2 \ldots \ell$$

for $N \geq 3$ where it is clear that $K > o$ may also be similarly defined for $N = 1$ and $N = 2$. In what follows we shall consider in detail only the case $N \geq 3$ since similar techniques would give the same results in the other two cases.

The integral of the square of the conormal derivative of ψ is obtained by bounding the unknown expressions on the right of 4.30 in terms of the given boundary data. From the divergence theorem and the differentiability of ψ and the coefficients of the differential operator, we have for an arbitrary constant "a"

4.63 $$\iiint_V \frac{\partial \tilde{\psi}}{\partial y} \mathfrak{F}(\tilde{\psi}) \exp\left[a(y_0 - y)\right] dv$$

$$= \iint_{D(y_0)} \tilde{\psi} F \, ds - \iint_{D(0)} G F \exp\left[(a+2K) y_0\right] ds$$

$$+ (a+K) \iiint_V \tilde{\psi} F \exp\left[(a+K)(y_0 - y)\right] dv$$

$$- \iiint_V \tilde{\psi} \frac{\partial F}{\partial y} \exp\left[(a+K)(y_0 - y)\right] dv \leq$$

$$\leq \left(-\frac{a}{2}+\frac{M_B}{2a_0}\right)\iiint_V a^{ij}\tilde{\varphi}_{,i}\tilde{\varphi}_{,j}\exp[a(y_0-y)]dv - \frac{1}{2}\iint_{D(y_0)} a^{ij}\hat{\varphi}_{,i}\hat{\varphi}_{,j}\,ds$$

$$+\frac{1}{2}\iint_{D(0)} a^{ij}G_{,i}G_{,j}\exp[(a+2k)y_0]ds + \frac{\varepsilon_1}{2}\iint_S\left(\frac{\partial\psi}{\partial\nu}\right)^2 ds$$

$$+\frac{1}{2\varepsilon_1}\iint_S\left[\frac{\partial H}{\partial y}-kH\right]^2\exp[2(a+2k)(y_0-y)]ds - \frac{1}{2}\iint_{D(y_0)}\bar{q}\,\tilde{\varphi}^2 ds$$

$$+\frac{1}{2}\iint_{D(0)}\bar{q}\,G^2\exp[(a+2k)y_0]ds - \frac{a}{2}\iiint_V \bar{q}\,\tilde{\varphi}^2\exp[a(y_0-y)]dv$$

$$+\frac{1}{2}\iiint_V\frac{\partial\bar{q}}{\partial y}\tilde{\varphi}^2\exp[a(y_0-y)]dv - \iiint_V z\left(\frac{\partial\tilde{\varphi}}{\partial y}\right)^2\exp[a(y_0-y)]dv.$$

The above inequality is transposed and use is made of 4.29, 4.39 c, 4.62 and Schwarz's inequality to obtain

4.64
$$\left\{\frac{a}{2}-\frac{M_B}{2a_0}-\frac{a}{2}\bar{m}-\frac{2M_{15}R^2}{N^2a_0}-\frac{2R^2}{N^2a_0}\right\}\iiint_V a^{ij}\hat{\varphi}_{,i}\hat{\varphi}_{,j}\,dv$$

$$+\left\{\frac{1}{2}-\frac{\bar{m}}{2}-\frac{2\varepsilon_2R^2}{N^2a_0}\right\}\iint_{D(y_0)} a^{ij}\varphi_{,i}\varphi_{,j}\,dv + \iiint_V z\left(\frac{\partial\tilde{\varphi}}{\partial y}\right)^2 dv$$

$$\leq \frac{\varepsilon_1}{2}\iint_S\left(\frac{\partial\psi}{\partial\nu}\right)^2 ds + \frac{1}{2\varepsilon_1}\iint_S\left[\frac{\partial H}{\partial y}-kH\right]^2\exp[2(a+2k)(y_0-y)]ds$$

$$+\frac{1}{2}\iint_{D(0)} a^{ij}G_{,i}G_{,j}\exp[(a+2k)y_0]ds + \frac{1}{2}\iint_{D(0)}\bar{q}\,G^2\exp[(a+2k)y_0]ds$$

$$+\frac{1}{2\varepsilon_2}\iint_{D(y_0)}F^2 ds + \iint_{D(0)} GF\exp[(a+2k)y_0]ds$$

$$+\frac{1}{2}\iiint_V\left[\frac{\partial F}{\partial y}-(a+k)F\right]^2\exp[2(a+k)(y_0-y)]dv.$$

In 4.30 and the last inequality, we set

4.65 $$\alpha_1 = \frac{m_3}{2}$$

4.66 $$\varepsilon_1 = 1$$

4.67 $$\varepsilon_2 = \frac{N^2 a_o}{4\underline{R}^2}(1 - \bar{m})$$

and

4.68 $$a = (1-\bar{m})^{-1}\left[\frac{M_8}{2a_0} + \frac{2M_{15}\underline{R}^2}{N^2 a_0} + \frac{2\underline{R}^2}{N^2 a_0} + \frac{2}{m_3}\left(C_1 + \frac{2M_1}{a_0} + \frac{2M_{13}}{a_0 m_3} + \frac{4M_{14}\underline{R}^2}{N^2 a_0}\right)\right]$$

Substitution of the above into 4.30 yields

4.69 $$\iint_S \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds \leq 2\mathcal{H} + \frac{8M_{14}}{m_3 N}\iint_S (x^i - \underline{x}^i)n_i \cdot H^2 \exp[2K(y_0 - y)]\,ds$$

$$+\frac{4}{m_3}\iiint_V F^2 \exp[2K(y_0-y)]\,dv + \iint_S \left[\frac{\partial H}{\partial y} - KH\right]^2 \exp[2(a+2K)(y_0-y)]\,ds$$

$$+\iint_{D(0)} a^{ij} G_{,i}\, G_{,j} \exp[(a+2K)y_0]\,ds + \iint_{D(0)} \bar{q}\, G^2 \exp[(a+2K)y_0]\,ds$$

$$+\frac{4\underline{R}^2}{N^2 a_0}(1-\bar{m})^{-1}\iint_{D(y_0)} F^2 ds + 2\iint_{D(0)} GF \exp[(a+2K)y_0]\,ds$$

$$+\iiint_V \left[\frac{\partial F}{\partial y} - (a+K)F\right]^2 \exp[2(a+K)(y_0-y)]\,dv = \mathcal{B}$$

where a is given by 4.68 and $K > o$ is chosen so that 4.62 is satisfied. Since $\mathcal{B}$ is a computable constant, the desired bound for the integral of the square of the conormal derivative has been obtained. A bound for the unknown volume integral in 4.17 is again given by 4.60, and thus the

desired pointwise bound is available.

2. The Dirichlet Problem at a Degenerate Point

We seek to obtain a bound for the solution of 4.1 at a degenerate point; that is, at a point p which is such that

4.70 $$z(p) = 0$$

At a degenerate point the author was unable to determine a parametrix with properties analogous to those of γ_p at a normally parabolic point. Because of this reason, additional conditions had to be placed on the boundary data, the shape of the domain and the region of validity of the differential operator.

Let the problem for which a bound for the solution is desired be that which is proposed at the beginning of the last section (specifically, from the heading to the paragraph immediately preceding equation 4.13) with 4.70 replacing 4.13. In addition we suppose that V is a cylindrical domain, h is twice piecewise continuously differentiable in y, and the differential equation is satisfied on D(o). Moreover we assume that the derivative with respect to y of the conormal derivative of w is continuous in the interior of a finite number of regions the sum of which is S and that the following are satisfied

4.71 $$\lim_{\substack{p_1 \to \bar{p} \in B(0) \\ p_1 \in D(0)}} g(p_1) = \lim_{\substack{p_2 \to \bar{p} \in B(0) \\ p_2 \in S}} h(p_2)$$

4.72 $\qquad z(x,o) > o$

4.73 $$\lim_{\substack{\bar{p}\to(x,o)\\ p\in V}} \frac{\partial w}{\partial y}(\bar{p}) = \frac{\partial w}{\partial y}(x,o)$$

and

4.74 $\qquad z(x,y_o) \equiv o$

for some open set $D^{*}\subset D(y_o)$ for which p is an interior point.

To shorten the argument, we consider only the case $N \geqslant 3$ since for N = 1 or N = 2 similar results are easily obtained. As in the previous sections, we obtain the desired bound by introducing a function $\psi = \psi(x,y)$ with the same differentiability as w and then with ψ given by 4.14 we compute the system 4.15. Next we define the function $\tilde{\psi}$ given by 3.20 and compute 4.19. The constant $K > o$ is chosen so that 4.21 c and

4.75 $$\min_{V} \left(\frac{1}{2}Kz + \frac{1}{2}\frac{\partial z}{\partial y}\right) > -(1-\bar{m})\frac{a_o N^2}{8\underline{R}^2}$$

are satisfied. The desired bound is now obtained by using (6.8) of [21] to write

4.76 $$\psi(p) = \iint_{D(y_0)} \left\{\Gamma_p \mathcal{a}(\psi) - \psi\mathcal{a}(\Gamma_p)\right\} ds$$

$$+ \oint_{B(y_0)} \left\{\psi \frac{\partial \Gamma_p}{\partial \nu} - \Gamma_p \frac{\partial \psi}{\partial \nu}\right\} d\ell$$

where

4.77 $$a(\psi) = (a^{ij}\psi_{,i})_{,j} - q\psi$$

$$= (a^{ij}\psi_{,j})_{,i} - q\psi$$

and Γ_p is given by (4.2) of [21]. The unknown terms on the right of 4.75 are the first, second and fourth. For the fourth term consider that

4.78 $$\oint_{B(y_0)} \Gamma_p \frac{\partial\psi}{\partial\nu} dl \le \left\{ \oint_{B(y_0)} \Gamma_p^2 \, dl \right\}^{\frac{1}{2}} \left\{ \oint_{B(y_0)} \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds \right\}^{\frac{1}{2}}.$$

Using Hörmander's Identity and the same techniques as were used to obtain 2.113, it is clear that

4.79 $$\oint_{B(y_0)} \left(\frac{\partial\psi}{\partial\nu}\right)^2 dl \le \bar{\mathcal{K}} + \frac{4}{\bar{m}_3} \iint_{D(y_0)} f^k \psi_{,k} (a^{ij}\psi_{,i})_{,j} \, ds$$

$$+ \frac{2\bar{C}_1}{\bar{m}_3} \iint_{D(y_0)} a^{ij} \psi_{,i} \psi_{,j} \, ds$$

where

4.80 $$\bar{\mathcal{K}} = \frac{4}{\bar{m}_3} \oint_{B(y_0)} (n f^k n_k)^{-1} \left(a_{ij} f^i t^j \frac{\partial\psi}{\partial t}\right)^2 dl$$

$$+ \frac{2}{\bar{m}_3} \oint_{B(y_0)} n^{-1} f^k n_k \left(\frac{\partial\psi}{\partial t}\right)^2 dl$$

4.81 $$\bar{m}_3 = \min_{B(y_0)} n^{-1} f^k n_k$$

and $f^i = f^i(x)$ are any set of N piecewise continuously

differentiable functions which are such that $f^i n_i$ is bounded and has a positive minimum on $B(y_0)$. From 4.32 after application of 4.50 and 4.47 or from 4.64 with the unknown term on the right bounded by 4.69, it is evident that one may obtain

4.82 $$\iint_{D(y_0)} a^{ij}\psi_{,i}\psi_{,j}\,ds \le \bar{\mathcal{D}}$$

where $\bar{\mathcal{D}}$ is a computable constant. Inequality 4.82 bounds the last term on the right of 4.79; for the second term we write

4.83 $$2\iint_{D(y_0)} f^k\psi_{,k}(a^{ij}\psi_{,i})_{,j}\,ds$$

$$= 2\iint_{D(y_0)} \bar{q} f^k\psi_{,k}\psi\,ds + 2\iint_{D(y_0)} z f^k\psi_{,k}\frac{\partial\tilde{\psi}}{\partial y}\,ds$$

$$+ 2\iint_{D(y_0)} F f^k\psi_{,k}\,ds$$

$$\le\left(\frac{2\bar{M}_1}{a_0} + \frac{\bar{M}_{13}}{a_0}\right)\iint_{D(y_0)} a^{ij}\psi_{,i}\psi_{,j}\,ds + \bar{M}_{14}\iint_{D(y_0)}\psi^2\,ds$$

$$+\iint_{D(y_0)} z\left(\frac{\partial\tilde{\psi}}{\partial y}\right)^2 ds + \iint_{D(y_0)} F^2\,ds$$

where

4.84 $$\bar{M}_1 = \max_{D(y_0)}\left\{\sum_1^N (f^i)^2\right\}$$

4.85 $$\bar{M}_{13} = \max_{D(y_0)}\left\{z\sum_1^N (f^i)^2\right\}$$

and

4.86 $$\bar{M}_{14} = \max_{D(y_o)} \left\{ \bar{q}^2 \right\}$$

By using an inequality over $D(y_o)$ which is analogous to 4.29, we may rewrite 4.79 as

4.87 $$\oint_{B(y_0)} \left(\frac{\partial\psi}{\partial\nu}\right)^2 dl \leq \mathcal{K}$$

$$+\frac{2}{\bar{m}_3}\left(\bar{C}_1 + \frac{2\bar{M}_1}{a_0} + \frac{\bar{M}_{13}}{a_0} + \frac{4\bar{M}_{14}\bar{R}^2}{N^2 a_0}\right)\bar{D} + \frac{4\bar{M}_{14}}{\bar{m}_3 N}\oint_{B(y_0)}(x^i - \underline{x}^i)m_i H^2 dl$$

$$+\frac{2}{\bar{m}_3}\iint_{D(y_0)} F^2 ds + \frac{2}{\bar{m}_3}\iint_{D(y_0)} z\left(\frac{\partial\tilde{\Psi}}{\partial y}\right)^2 dV$$

$$= \mathcal{M}_5 + \frac{2}{\bar{m}_3}\iint_{D(y_0)} z\left(\frac{\partial\tilde{\Psi}}{\partial y}\right)^2 dV$$

For the last term on the right we use the divergence theorem as follows:

4.88 $$\iiint_V \frac{\partial\tilde{\Psi}}{\partial y}\frac{\partial\tilde{J}}{\partial y} dV = -\iiint_V \frac{\partial\tilde{\Psi}_{,i}}{\partial y} a^{ij}\frac{\partial\tilde{\Psi}_{,j}}{\partial y} dV - \iiint_V \frac{\partial\tilde{\Psi}_{,i}}{\partial y}\frac{\partial a^{ij}}{\partial y}\tilde{\Psi}_{,j} dV$$

$$+\iint_S \frac{\partial\tilde{\Psi}}{\partial y}\frac{\partial}{\partial y}\frac{\partial\tilde{\Psi}}{\partial\nu} ds - \iiint_V \bar{q}\left(\frac{\partial\tilde{\Psi}}{\partial y}\right)^2 dV - \frac{1}{2}\iint_{D(y_0)} z\left(\frac{\partial\tilde{\Psi}}{\partial y}\right)^2 ds$$

$$+\frac{1}{2}\iint_{D(0)} z\left(\frac{\partial\tilde{\Psi}}{\partial y}\right)^2 ds - \frac{1}{2}\iiint_V \left(\frac{\partial\tilde{\Psi}}{\partial y}\right)^2 \frac{\partial z}{\partial y} dV.$$

Since

4.89 $$\left(\psi_{,i}\frac{\partial a^{ij}}{\partial y}\frac{\partial \psi_{,j}}{\partial y}\right)^2 \le \sum_{i=1}^{N}\psi_{,i}^{\,2}\sum_{r=1}^{N}\left(\frac{\partial a^{rj}}{\partial y}\frac{\partial \psi_{,j}}{\partial y}\right)^2$$

$$\le\left(\sum_{i=1}^{N}\psi_{,i}^{\,2}\right)\left(\sum_{r,s=1}^{N}\left[\frac{\partial a^{rs}}{\partial y}\right]^2\right)\left(\sum_{j=1}^{N}\left[\frac{\partial \psi_{,j}}{\partial y}\right]^2\right)$$

it follows that

4.90 $$\psi_{,i}\frac{\partial a^{ij}}{\partial y}\frac{\partial \psi_{,j}}{\partial y} \le \frac{M_8^2}{2\alpha_1 a_0}a^{ij}\psi_{,i}\psi_{,j} + \frac{\alpha_1}{2}a^{ij}\frac{\partial \psi_{,i}}{\partial y}\frac{\partial \psi_{,j}}{\partial y}.$$

From 4.47 after application of 4.50 or from 4.64 with the unknown term bounded by 4.69, one has

4.91 $$\iiint_V a^{ij}\tilde{\psi}_{,i}\tilde{\psi}_{,j}\,dv \le \mathcal{D}.$$

By transposing 4.88, and integrating by parts, we obtain

4.92 $$\frac{1}{2}\iint_{D(y_0)} z\left(\frac{\partial\tilde{\psi}}{\partial y}\right)^2 ds + \left\{\frac{1-\bar{m}}{2}-\frac{\alpha_1}{2a_0}-\frac{2\alpha_2 R^2}{a_0 N^2}\right\}\iiint_V \frac{\partial\tilde{\psi}_{,i}}{\partial y}a^{ij}\frac{\partial\tilde{\psi}_{,j}}{\partial y}dV$$

$$\le + \frac{M_8^2}{2\alpha_1 a_0}\mathcal{D} + \frac{\alpha_3}{2}\oint_{B(y_0)}\left(\frac{\partial\psi}{\partial\nu}\right)^2 d\ell + \frac{1}{2\alpha_3}\oint_{B(y_0)}\left[-KH+\frac{\partial H}{\partial y}\right]^2 d\ell$$

$$+\frac{\alpha_2}{N}\iint_S (x^i-\underline{x}^i)n_i\left[-KH+\frac{\partial H}{\partial y}\right]^2 \exp[2K(y_0-y)]\,ds$$

$$+\frac{1}{2\alpha_2}\iiint_V\left[-KF+\frac{\partial F}{\partial y}\right]^2\exp[2K(y_0-y)]\,dV + \mathcal{M}_6$$

where

4.93 $$\mathcal{M}_6 = -\oint_{B(0)} \frac{\partial G}{\partial \nu}\left\{-KH + \frac{\partial H}{\partial y}\right\} \exp(2Ky_0)\, d\ell$$

$$+\frac{1}{2}\iint_S \left\{K^2 H - 2K\frac{\partial H}{\partial y} + \frac{\partial^2 H}{\partial y^2}\right\}^2 \exp[2K(y_0 - y)]\, ds + \frac{1}{2}\mathcal{B}$$

$$-\sum_{I=1}^{d} \varepsilon^I \tilde{m}_I \left(\frac{2}{N-2}\right) \iint_S (x^i - x^i_I) r_I^{-2} n_i \left\{-KH + \frac{\partial H}{\partial y}\right\}^2 \exp[2K(y_0 - y)]\, ds$$

$$+\frac{1}{2}\iint_{D(0)} z^{-1}\left\{(a^{ij}G_{,i})_{,j} - qG - F\right\}^2 \exp(2Ky_0)\, ds$$

$$+(1-\bar{m})\frac{q_0 N}{4\beta^2}\iint_S (x^i - \underline{x}^i) n_i \left\{-KH + \frac{\partial H}{\partial y}\right\}^2 \exp[2K(y_0 - y)]\, ds.$$

In 4.92 we choose the strictly positive numbers α_1 and α_2 so that the coefficient of the second term on the left is zero. By comparing 4.87 and 4.92 we are led to define

4.94 $$\alpha_3 = \frac{1}{4}\bar{m}_3$$

By fixing the arbitrary positive constants as indicated above, it is clear that one easily obtains

4.95 $$\oint_{B(y_0)} \left(\frac{\partial \psi}{\partial \nu}\right)^2 d\ell \le \bar{\mathcal{B}}$$

where $\bar{\mathcal{B}}$ is a computable constant. Inequalities 4.95 and 4.78 provide the desired estimate for the last term on the right of 4.75. The second term on the right of that equation is bounded by using exactly the same procedures as Payne and Weinberger employed in [21]. Let $\mathcal{F}(p)$ be defined by (4.18) of [21]; it is then clear that ρ_0 and b

may be chosen so that

4.96 $$-\mathcal{F} \leq \bar{M} \qquad \text{for } (x,y) \in D(y_0)$$

and

4.97 $$a^*(\mathcal{F}) < 0 \qquad \text{for } (x,y) \in D(y_0)$$

where

4.98 $$a^*(\mathcal{F}) = (a^{ij}\mathcal{F}_{,j})_{,i} - 2q\mathcal{F}$$

From the divergence theorem and the definition of $\mathcal{F}$

4.99
$$\iint_{D(y_0)} \{\mathcal{F}a(\psi^2) - \psi^2 a^*(\mathcal{F})\}ds - \iint_{D(y_0)}\{q\mathcal{F}\psi^2 + 2\mathcal{F}a^{ij}\psi_{,i}\psi_{,j}\}ds$$
$$= -\iint_{D(y_0)} \psi^2 a^*(\mathcal{F})ds + 2\iint_{D(y_0)} \psi\mathcal{F}a(\psi)ds$$
$$\leq \oint_{B(y_0)} \mathcal{F}^2 H^2 d\ell + \bar{\mathcal{B}} - \oint_{B(y_0)} H^2 \frac{\partial\mathcal{F}}{\partial\nu}d\ell + 2\bar{M}\bar{\mathcal{D}} = \bar{\mathcal{J}}$$

and thus

4.100
$$\left\{\iint_{D(y_0)} -\psi^2 a^*(\mathcal{F})\,ds\right\}^{\frac{1}{2}}$$
$$\leq \left\{\iint_{D(y_0)} [-a^*(\mathcal{F})]^{-1}\mathcal{F}^2 a(\psi)^2 ds\right\}^{\frac{1}{2}}$$
$$+\left\{\iint_{D(y_0)} [-a^*(\mathcal{F})]^{-1}\mathcal{F}^2 a(\psi)^2 ds + \bar{\mathcal{J}}\right\}^{\frac{1}{2}}.$$

To bound the unknown term in the above expression we make use of 4.82 as follows

4.101 $$\iint_{D(y_0)} \psi^2 ds \le \frac{2}{N}\oint_{B(y_0)} (x^1 - \underline{x}^1) n_1 H^2 ds + \frac{4R^2}{N^2 a_0}\bar{\mathcal{D}} = m_7$$

and then obtain

4.102 $$\iint_{D(y_0)} [-a^*(\mathcal{F})]^{-1}\mathcal{F}^2 a(\psi)^2 ds$$

$$\le 2M_{16}\iint_{D(y_0)} z\left(\frac{\partial\psi}{\partial y}\right)^2 ds + 2\iint_{D(y_0)} [-a^*(\mathcal{F})]^{-1}\mathcal{F}^2 F^2 ds$$

$$\le 4\, M_{16} M_{17}\, m_7 + 4\, M_{16}\, m_8 + 2\iint_{D(y_0)} [-a^*(\mathcal{F})]^{-1}\mathcal{F}^2 F^2 ds$$

where

4.103 $$M_{16} = \max_{D(y_0)-D^*} \left\{ [-a^*(\mathcal{F})]^{-1}\mathcal{F}^2 z \right\}$$

4.104 $$M_{17} = \max_{D(y_0)-D^*} \left\{ zK^2 \right\}$$

and from 4.92 after application of 4.95

4.105 $$\iint_{D(y_0)} z\left(\frac{\partial\Psi}{\partial y}\right)^2 ds \le m_8$$

By means of Schwarz's inequality

4.106 $$\left|\iint_{D(y_0)} \psi\, a(\Gamma_p)\, ds\right|$$

$$\le \left\{\iint_{D(y_0)} -\psi^2 a^*(\mathcal{F})\right\}^{1/2} \left\{\iint_{D(y_0)} [-a^*(\mathcal{F})]^{-1} a(\Gamma_p)^2 ds\right\}^{1/2}$$

In the above the first factor on the right is bounded by 4.100 and 4.102 while the second may be computed due to

(4.1 b) of [21]. To estimate the first term on the right of 4.75 we consider

4.107
$$\iint_{D(y_0)} \Gamma_p a(\psi)\, ds$$
$$= \iint_{D(y_0)} z\Gamma_p \frac{\partial \psi}{\partial y}\, ds + \iint_{D(y_0)} \Gamma_p F\, dv$$
$$\leq 2 M_{18} M_{17} m_7 + 2 M_{18} m_8 + \iint_{D(y_0)} \Gamma_p F\, dv$$

where

4.108
$$M_{18} = \max_{D(y_o)-D^*} \{z\, \Gamma_p^2\}.$$

Transposing 4.75 and taking the absolute value yields

4.109
$$\left| W(p) - \varphi(p) - \iint_{D(y_0)} \Gamma_p F\, ds - \oint_{B(y_0)} H \frac{\partial \Gamma_p}{\partial \nu}\, d\ell \right|$$
$$\leq 2 M_{18} M_{17} m_7 + 2 M_{18} m_8 + \left| \iint_{D(y_0)} \psi\, a(\Gamma_p)\, ds \right|$$
$$+ \left| \oint_{B(y_0)} \Gamma_p \frac{\partial \psi}{\partial \nu}\, d\ell \right|$$

where 4.106 and 4.78 bound the last two expressions in terms of computable constants.

It is the belief of the author that the conditions imposed on the boundary value problem of this section are excessive. In future investigations, efforts shall be made to produce pointwise bounds at a degenerate point

without imposing the rather restrictive conditions which were necessary in order that the above techniques yield the desired result.

3. The Mixed Problem at a Parabolic Point

Thus far in this chapter we have always required that the matrix a^{ij} be positive definite and that z be a non-negative function. Now we suppose that z is strictly positive but reduce the conditions on the matrix by requiring that a^{ij} be positive semi-definite.

Let V be an (N + 1) dimensional domain and let the notation of the problem be as indicated in equations 2.1 to 2.4. In place of condition 2.5 we now require that

4.110 $$\min_S n_y = m_5 > 0$$

We shall derive pointwise bounds for the solution of

4.111 $$\begin{cases} J(w) = (a^{ij} w_{x_i})_{x_j} - z \dfrac{\partial w}{\partial y} = f(x,y,w) & (x,y) \in V \\ w(x,0) = g(x) & (x,y) \in D(0) \\ \dfrac{\partial w}{\partial \nu} = \ell(x,y) & (x,y) \in S \end{cases}$$

where clearly it is necessary that

4.112 $$\ell(x,y) = 0$$

for $(x,y) \in S_1$ with

4.113 $$S_1 = \left\{ (x,y) \,\middle|\, (x,y) \in S,\ \sum_1^N n_i^2 = 0 \right\}.$$

The solution function w, the coefficients of the differential operator, and the boundary data have the same differentiability and integrability as the analogous functions of the mixed problem of Chapter III. As in the previous problem, the function f satisfies a Lipschitz condition in w, that is,

4.114 $$\left| f(x,y,w_1) - f(x,y,w_2) \right| \le \tilde{M}_1 \left| w_1 - w_2 \right| .$$

As the title implies, the solution is to be bounded at a parabolic point $p \in D(y_0)$. Thus a^{ij} is positive definite so that $a_{ij}(p)$ exists. In addition to the above it is necessary to assume that conditions 2.8, 2.9, and 3.5 are satisfied at p.

We use the notation and techniques of the previous mixed problem to obtain the following equation which is analogous to 3.16

4.115 $$\left| w(x_0,y_0) - \chi(p) \right| \le \left| \lim_{y \to y_0} \iiint_{V(y)} \psi \bar{J}(\gamma_p)\, dV \right|$$
$$+ \left| \lim_{y \to y_0} \iiint_{V(y)} \gamma_p \{ f(x,y,w) - f(x,y,\varphi) \}\, dV \right|$$
$$+ \left\{ \iint_S \psi^2 ds \right\}^{1/2} \left\{ \iint_S \left(\frac{\partial \gamma_p}{\partial \nu} + n_y \mp \gamma_p \right)^2 ds \right\}^{1/2}$$

where

4.116 $$\chi(p) = \varphi(x_0, y_0) + \iint_S L\gamma_p \, ds$$

$$+ \iint_{D(0)} \gamma_p z G \, ds \; - \lim_{y \to y_0} \iiint_{V(y)} \gamma_p F \, dV.$$

The right side of 4.116 is computable by the arguments associated with equations 3.18 and 3.19; and hence the desired bound is available when the unknown expressions on the right of 4.115 have been estimated. To estimate the boundary integral the change of variables given by 3.20 to 3.22 is made and then by means of the divergence theorem it is clear that

4.117 $$\iint_S \tilde{\varphi}^2 ds \le 2(\underline{m}_2 m_5)^{-1} \Big\{ \frac{\alpha_1}{2} \iint_S \tilde{\varphi}^2 ds$$

$$+ \frac{1}{2\alpha_1} \iint_S L^2 \exp[2K(y_0 - y)] ds + (-Km_2 + \tilde{M}_1 + \tfrac{1}{2} + \tfrac{1}{2} M_4) \iiint_V \tilde{\varphi}^2 dV$$

$$+ \frac{1}{2} \iiint_V F^2 \exp[2K(y_0 - y)] dV + \frac{1}{2} \iint_{D(0)} z G^2 \exp(2Ky_0) \, ds \Big\}$$

where

4.118 $$\min_S z = \underline{m}_2 \geqslant m_2 > 0$$

In order to obtain a non-trivial bound for the left side of 4.117 we set

4.119 $$\alpha_1 = \tfrac{1}{2}(\underline{m}_2 m_5)$$

and

4.120 $$K = m_2^{-1}\left(\tilde{M}_1 + \tfrac{1}{2} + \tfrac{1}{2} M_4\right) = K_1$$

Due to the above we may write

4.121 $$\iint_S \psi^2 \, ds \leq 4 (\underline{m}_2 m_5)^{-2} \iint_S L^2 \exp[2K_1(y_0 - y)] \, ds$$

$$+ 2 (\underline{m}_2 m_5)^{-1} \iiint_V F^2 \exp[2K_1(y_0 - y)] \, dv$$

$$+ 2 (\underline{m}_2 m_5)^{-1} \iint_{D(0)} \mathfrak{z} G^2 \exp(2K_1 y_0) \, ds$$

$$= \mathcal{B}$$

In order to bound the volume integrals of 4.115 we introduce the function $\mathcal{F}$ given by 2.39. Even though the differential operator of this section does not satisfy the conditions of Theorem 2.2, an inspection of the proof reveals that the conclusions of the theorem are valid for the present problem. First we observe that constants ρ_0, $\delta_{\sigma 1}$, and $\delta_{\sigma 2}$ associated with equations 2.36 and 2.37 exist since J is normally parabolic at the point p and thus a^{ij} is positive definite in a small neighborhood of p due to 2.8. Because $\mathfrak{z}$ is strictly positive in $\overline{V}$ it easily follows by remarks similar to those which were employed in the proof of Theorem 2.2 that the constants

α and ℓ of 2.39 may be chosen so that

4.122 $\qquad \bar{J}(\mathcal{F}) \leq 0, \; \mathcal{F} \leq 0 \qquad\qquad (x,y) \in \bar{V}$

and

4.123 $\qquad \bar{J}(\mathcal{F}) < 0, \mathcal{F} < 0, \; (x,y) \in \{x,y \,|\, (x,y) \in \bar{V}, 0 < \varepsilon \leq y_0 - y\}.$

Due to the foregoing, equations 3.49 to 3.75 are valid for the present problem with

4.124 $\qquad \tilde{M}_2 = \tilde{M}^1 = 0.$

The bounds for the unknown volume integrals of 4.115 are hence available by means of the appropriate equations of Chapter III. After application of these results one obtains

4.125 $$|w(x_0,y) - \chi(p)|$$

$$\leq \left\{ \iiint_V [-\bar{J}(\mathcal{F})]^{-1} \bar{J}(\gamma_p)^2 \, dv \right\}^{1/2}$$

$$\cdot \left\langle \left\{ \iiint_V [-\bar{J}(\mathcal{F})]^{-1} (\mathcal{F} F)^2 \exp[2k_2(y_0 - y)] \, dv \right\}^{1/2} \right.$$

$$\left. + \left\{ \mathcal{J}_2 + \iiint_V [-\bar{J}(\mathcal{F})]^{-1} (\mathcal{F} F)^2 \exp[2k_2(y_0 - y)] \, dv \right\}^{1/2} \right\rangle$$

$$+ \left\{ \iiint_V \gamma_p^2 \mathcal{F}^{-1} \exp[2k_3(y_0 - y)] \, dv \right\}^{1/2} \{\tilde{\mathcal{J}}\}^{1/2}$$

$$+ \left\{ \iint_S \left(\frac{\partial \gamma_p}{\partial \nu} + n_y z \gamma_p \right)^2 ds \right\}^{1/2} \{\mathcal{B}\}^{1/2}$$

where γ_p is given by 2.10 and $\mathcal{F}$ is defined by 2.39 subject to the remarks of this section. Using the results of Chapter III we obtain that K_2 and K_3 are given by 3.62 and 3.74a while the computable constants $\mathcal{J}_2$ and $\tilde{\mathcal{J}}$ are defined by 3.57 and 3.75a. The bound $\mathcal{B}$ is determined by the methods of this section and is given explicitly by 4.121.

CHAPTER V

THE ELLIPTIC PROBLEM

1. The Neumann Problem

In [21], Payne and Weinberger obtained pointwise bounds for the solution of second order interior elliptic differential equations with Dirichlet or mixed boundary data. The occurrence of the zero eigenvalue of the free membrane problem prevented the authors from extending their results to Neumann boundary value problems. By following closely the procedures which were used to obtain a bound for the solution of a normally parabolic mixed problems, we obtain such bounds for an elliptic problem of the same general type.

In what follows we shall use the notation and many of the results of section 6 of [21]. We seek pointwise bounds for the solution of

$$B(w) = (a^{ij}w_{,i})_{,j} - b^{i}w_{,i} - qw = f(x) \qquad x \in D$$

5.1

$$\frac{\partial w}{\partial \nu} = \ell(x) \qquad x \in B$$

where D is a domain of Euclidean N-space with boundary B. In order that the divergence theorem be applicable it is necessary to assume that w is continuous in $\overline{D}$, continuously

differentiable in D and is such that its second derivative is continuous in the interior of a finite number of subregion, the sum of which is D. In addition it is necessary to assume that the solution of the boundary value problem is such that

5.2 $$\lim_{\substack{\bar{p} \to x \in B \\ \bar{p} \in D}} a^{ij}(\bar{p}) w_{,j}(\bar{p}) = a^{ij}(x) w_{,j}(x) \qquad i = 1, 2 \ldots N$$

We suppose that the components of the matrix a^{ij} and the b^i are piecewise continuously differentiable in $\bar{D}$ while $q(x)$ is bounded and strictly positive in the same domain. As in (6.3) of [21] it is necessary that

5.3 $$0 \le b_0 \left[\sum_1^N \xi_i^2 + q(x)\, \xi_0^2 \right] \le a^{ij} \xi_i \xi_j + b^i \xi_0 \xi_i + q(x)\, \xi_0^2$$

$$\le b_1 \left[\sum_1^N \xi_i^2 + q(x)\, \xi_0^2 \right]$$

for all real numbers $(\xi_0 \ldots \xi_N)$ where equality holds on the left if and only if all the numbers are zero. Let

5.4 $$\min_D q(x) = m_1 > 0$$

where m_1 is positive since $q(x)$ is strictly positive by assumption. From (6.8) of [21] we have

5.5 $$W(p) - \varphi(p) = \iint_D \left\{ \Gamma_p B(w-\varphi) - (w-\varphi) \bar{B}(\Gamma_p) \right\} dv$$

$$+ \oint_B \left\{ (w-\varphi) \left(\frac{\partial \Gamma_p}{\partial \nu} + b^i n_i \Gamma_p \right) - \Gamma_p \frac{\partial}{\partial \nu} (w-\varphi) \right\} ds$$

where

5.6 $$\bar{B}(\Gamma_p) = (a^{ij}\Gamma_{p,j})_{,i} + (b^i\Gamma_p)_{,i} - q\Gamma_p$$

with Γ_p given by (4.2) of [21] and φ defined as in the second paragraph of section 4 of that reference. We apply Schwarz's inequality to the first boundary term in 5.5 and obtain

5.7 $$\oint_B \psi\left(\frac{\partial\Gamma_p}{\partial\nu} + b^i n_i \Gamma_p\right)ds \le \left\{\oint_B \psi^2 ds\right\}^{1/2}\left\{\oint_B\left(\frac{\partial\Gamma_p}{\partial\nu} + b^i n_i\Gamma_p\right)^2 ds\right\}^{1/2}$$

where, as in the previous parts of this thesis, we have set

5.8 $$\psi = w - \varphi.$$

Let $f^1(x) \ldots f^N(x)$ be the set of functions introduced in sections 2 and 3 of the work by Payne and Weinberger, that is, the set $f^i(x)$ is piecewise continuously differentiable in D and is such that $f^k n_k$ is bounded and has a positive minimum on B. For later use, we set

5.9 $$m_2 = \min_B f^k n_k$$

and then bound the unknown factor on the right of 5.7 by writing

5.10 $$\oint_B \psi^2 ds \le \frac{1}{m_2}\oint_B \psi^2 f^i n_i\, ds = \frac{1}{m_2}\iint_D [\psi^2 f^i]_{,i}\, dv$$

$$= \frac{2}{m_2}\iint_D \psi_{,i}\psi f^i dv + \frac{1}{m_2}\iint_D \psi^2 f^i_{,i}\, dv$$

$$\le \frac{1}{m_2}(\varepsilon_1 + M_1)\iint_D \psi^2 dv + \frac{M_2}{m_2\varepsilon_1}\iint_D \sum_1^N \psi_{,i}^2\, dv$$

where

5.11 $$M_1 = \max_D \left\{ f^i_{,i} \right\}$$

5.12 $$M_2 = \max_D \left\{ \sum_1^N f_i^2 \right\}$$

and ε_1 is an arbitrary positive number. By means of the boundary value problem and the divergence theorem, it is evident that

5.13 $$\iint_D \psi F \, dv = -\iint_D \psi_{,i} a^{ij} \psi_{,j} \, dv + \oint_B \psi L \, ds$$

$$-\iint_D \psi b^i \psi_{,i} \, dv - \iint_D q \psi^2 \, dv$$

where

5.14 $$F(x) = f(x) - B(\varphi)$$

and

5.15 $$L(x) = \ell(x) - \frac{\partial \varphi}{\partial \nu} .$$

By means of 5.3 and a transposed form of 5.13, the following holds for an arbitrary positive numbers ε_2 and ε_3

5.16
$$b_0 \iint_D \Big\{\sum_1^N \psi_{x_i}^2 + \big(m_1 - \tfrac{\varepsilon_3}{2b_0}\big)\psi^2\Big\} dv$$

$$\le \iint_D a^{ij}\psi_{x_i}\psi_{x_j}\,dv + \iint_D b^i \psi_{x_i}\psi\,dv + \iint_D \big(q - \tfrac{\varepsilon_3}{2}\big)\psi^2 dv$$

$$\le \frac{1}{2\varepsilon_3}\iint_D F(x)^2 dv + \frac{\varepsilon_2}{2}\oint_B \psi^2 ds + \frac{1}{2\varepsilon_2}\oint_B L(x)^2 ds\,.$$

In order to obtain a non-trivial bound we choose

5.17
$$\varepsilon_3 = m_1 b_0\,;$$

and hence

5.18
$$\iint_D \Big\{\sum_1^N \psi_{x_i}^2 + \frac{m_1}{2}\psi^2\Big\} dv$$

$$\le \frac{1}{2m_1 b_0^2}\iint_D F(x)^2 dv + \frac{\varepsilon_2}{2b_0}\oint_B \psi^2 ds + \frac{1}{2\varepsilon_2 b_0}\oint_B L(x)^2 ds\,.$$

So that the last equation may be employed to bound 5.10 we rewrite the previous equation as follows

5.19
$$\oint_B \psi^2 ds \le \frac{2}{m_1 m_2}(\varepsilon_1 + M_1)\iint_D \frac{m_1}{2}\psi^2 dv + \frac{M_2}{m_2\varepsilon_1}\iint_D \sum_1^N \psi_{x_i}^2\,dv$$

and then choose ε_1 so that the coefficients on the right are equal. We solve the equation

5.20
$$2\varepsilon_1^2 + 2M_1\varepsilon_1 - m_1 M_2 = 0$$

and obtain

5.21 $$\varepsilon_1 = 4^{-1}\left[-2M_1 \pm \sqrt{4M_1^2 + 8m_1M_2}\right]$$

Since ε_1 is to be a strictly positive number, the positive square root is chosen and thus

5.22 $$\varepsilon_1 = \frac{1}{2}\,(-M_1 + \sqrt{M_1^2 + 2m_1M_2})\,.$$

To shorten the notation we let

5.23 $$M_3 = \frac{2}{m_1M_2}\left[\frac{1}{2}\,(-M_1 + \sqrt{M_1^2 + 2m_1M_2}) + M_1\right]$$

and bound 5.19 with 5.18 as follows

5.24 $$\left(1 - \frac{M_3\varepsilon_2}{2b_0}\right)\oint_B \psi^2 ds \le \frac{M_3}{2m_1b_0^2}\iint_D F^2 dV + \frac{M_3}{2\varepsilon_2 b_0}\oint_B L^2 ds\,.$$

Since a non-trival bound of the left side of the last equation is desired we choose the arbitrary number ε_2 in an appropriate manner:

5.25 $$\varepsilon_2 = \frac{b_0}{M_3}$$

Substitution of 5.25 into 5.24 yields

5.26 $$\oint_B \psi^2 ds \le \frac{M_3}{m_1b_0^2}\iint_D F^2 dV + \frac{M_3^2}{b_0^2}\oint_B L^2 ds = \mathcal{B}$$

which gives an estimate for the unknown boundary term of 5.5 when this term is first bounded by 5.7.

To bound the unknown volume term of 5.5 the following cases are considered. For N = 2 or N = 3 the results of pages 562 and 563 of [21] are applicable since from 5.18 and 5.26 we have

5.27 $$\iint_D \sum_1^N \psi_{,i}^2 \, dv \leq \frac{1}{2m_1 b_0^2} \iint_D F^2 dv + \frac{1}{b_0} \{\mathcal{B}\}^{\frac{1}{2}} \{\oint_B L^2 ds\}^{\frac{1}{2}} = \mathcal{O}$$

For $N > 3$ a function $\mathcal{F}$ is defined by (4.18) of [21] so that

5.28 $$-\mathcal{F} \leq M \qquad x \in D$$

and

5.29 $$\bar{B}(\mathcal{F}) < 0 \qquad x \in D.$$

From the divergence theorem, it can be shown that

5.30 $$+\iint_D \{2\mathcal{F}\psi B(\psi) - \psi^2 \bar{B}(\mathcal{F})\} dv \leq$$

$$\leq 2 M b_1 \mathcal{O} + 2\{\mathcal{B}\}^{\frac{1}{2}} \{\oint_B \mathcal{F}^2 L^2 ds\}^{\frac{1}{2}}$$

$$+ M_4 \mathcal{B} = \mathcal{J}$$

where

5.31 $$M_4 = \max_B \left\{ \frac{\partial \mathcal{F}}{\partial \nu} + b^i n_i \mathcal{F} \right\}.$$

Hence

5.32 $$\{-\iint_D \psi^2 \bar{B}(\mathcal{F}) dv\}^{\frac{1}{2}}$$

$$\leq \{\iint_D [-\bar{B}(\mathcal{F})]^{-1} \mathcal{F}^2 F^2 dv\}^{\frac{1}{2}}$$

$$+ \{\iint_D [-\bar{B}(\mathcal{F})]^{-1} \mathcal{F}^2 F^2 dv + \mathcal{J}\}^{\frac{1}{2}}.$$

The desired result is now obtained by applying Schwarz's inequality to the unknown volume integral which we are seeking to bound. We obtain

5.33 $$\left|\iint_D \psi^2 \bar{B}(\Gamma_p)\,dv\right| \leq \left\{-\iint_D \bar{B}(\mathcal{F})\psi^2 dv\right\}^{1/2}\left\{\iint_D [-\bar{B}(\mathcal{F})]^{-1}[\bar{B}(\Gamma_p)]^2 dv\right\}^{1/2}$$

and note that the first factor on the right is bounded by 5.32 while the second is computable due to (4.1) of [21].

From the above it is clear that

5.34 $$|W(p)-\chi(p)| \leq \{\mathcal{B}\}^{1/2}\left\{\oint_B \left(\frac{\partial \Gamma_p}{\partial \nu} + b^i n_i \Gamma_p\right)^2 ds\right\}^{1/2} + \left|\iint_D \psi^2 \bar{B}(\Gamma_p)\,dv\right|$$

where $\mathcal{B}$ is given by 5.26, the second term on the right is bounded by 5.33 and

5.35 $$\chi(p) = -\oint_B \Gamma_p L\,ds + \iint_D \Gamma_p F\,dv + \varphi(p) .$$

2. A Generalized Dirichlet Problem

In this section a pointwise bound for the solution of a Dirichlet problem shall be obtained where the given differential operator is such that the results of [21] are not applicable.

Since the problem and the techniques which are to be employed are very similar to those which were used for the Degenerate Dirichlet Problem, the notation of [21] shall be

altered so that it more closely follows that which was previously developed in this paper.

We seek a pointwise bound for the solution of

5.36
$$\begin{cases} J(w) = (a^{ij}w_{,i})_{,j} - qw = f(x) & x \in D \\ w(x) = h(x) & x \in B \end{cases}$$

where D and B are as defined in the first section of this chapter. The solution function w is assumed to be continuously differentiable in $\bar{D}$ and to have a continuous second derivative in the interior of a finite number of regions the sum of which is D. The symmetric matrix a^{ij} is piecewise continuously differentiable and satisfies

5.37
$$0 \leq a_0 \sum_{1}^{N} \xi_i^2 \leq a^{ij}\xi_i \xi_j \leq a_1 \sum_{1}^{N} \xi_i^2$$

for all real numbers $(\xi_1, \ldots \xi_N)$ where equality holds on the left if and only if all the numbers are zero. The bounded integrable function $q = q(x)$ is very similar to the analogous function of the Degenerate Dirichlet Problem; that is, q has the form

5.38
$$q = \varepsilon^I q_I = \sum_{I=1}^{\ell} \varepsilon^I q_I$$

where

5.39
$$\sum_{I=1}^{\ell} \varepsilon^I = 1$$

and for $N \geqslant 3$

5.40 $$\tilde{m}_I = \min_D \{0 \,;\, r_I^2\, q_I > -\tfrac{1}{4} q_o (N-2)^2\} \qquad I = 1, 2, \ldots \ell$$

Conditions similar to 4.8 a and 4.8 b may easily be obtained for N = 1 and N = 2. To shorten the argument, we consider in complete detail only the case $N \geqslant 3$. As for the degenerate problem, we define $\bar{m}$ as

5.41 $$0 \leqslant -4\, \varepsilon^I \tilde{m}_I \left[a_o (N-2)^2\right]^{-1} = \bar{m} < 1 .$$

So that the divergence theorem and Schwarz's inequality are applicable we assume that f is integrable and square integrable over D.

To obtain the desired bound for the solution at an interior point $p \in D$ we choose an arbitrary function $\varphi = \varphi(x)$ which is such that $J(\varphi)$ and $\varphi(x)$ approximate f(x) and h(x) as appropriate. We further require that φ be twice piecewise continuously differentiable in $\bar{D}$. We introduce the function

5.42 $$\psi(x) = w(x) - \varphi(x)$$

and compute

5.43 $$\begin{cases} J(\psi) = F(x) & x \in D \\ \psi(x) = H(x) & x \in B . \end{cases}$$

From (6.8) of [21] we have

5.44 $$w(p) - \varphi(p)$$

$$= \iiint_D \{\Gamma_p\, J(\psi) - \psi\, \bar{J}(\Gamma_p)\} dv + \oint_B \{\psi \frac{\partial \Gamma_p}{\partial \nu} - \Gamma_p \frac{\partial \psi}{\partial \nu}\} ds$$

where

5.45 $$\bar{J}(\Gamma_p) = (a^{ij}\Gamma_{p,i})_{,j} - q\Gamma_p$$

with Γ_p given by (4.2) of the work by Payne and Weinberger. The unknown terms on the right of 5.44 are the second and fourth. The final term is the first to be estimated. With B replacing $B(y_o)$ and D replacing $D(y_o)$ we note that equations 4.78 to 4.81 are valid for the present problem. Next we use

5.46 $$\iint_D \psi^2 dv \le \frac{2}{N}\oint_B (x^i - \underline{x}^i) n_i \psi^2 ds + \frac{4\underline{R}^2}{N^2 a_0}\iint_D a^{ij}\psi_{,i}\psi_{,j}\, dv$$

to write

5.47 $$\oint_B \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds \le \bar{\mathcal{H}} + \frac{2}{\bar{m}_3}\left(\bar{C}_1 + \frac{2\bar{M}_1}{a_0} + \frac{4\bar{M}_{14}\underline{R}^2}{N^2 a_0}\right)\iint_D a^{ij}\psi_{,i}\psi_{,j}\, dv$$
$$+ \frac{4\bar{M}_{14}}{\bar{m}_3 N}\oint_B (x^i - \underline{x}^i) n_i H^2 ds + \frac{2}{\bar{m}_3}\iint_D F^2 ds$$

where

5.48 $$\bar{m}_3 = \min_B n^{-1} f^k n_k$$

5.49 $$\underline{R}^2 = \min_{\underline{x}\in D}\max_{x\in B}\sum_1^N (x^i - \underline{x}^i)^2$$

5.50 $$\bar{M}_1 = \max_D \sum_1^N (f^i)^2$$

5.51 $$\bar{M}_{14} = \max_D \left\{ q^2 \right\}$$

and, as indicated above, $\bar{\mathcal{K}}$ is given by 4.80. To determine a bound for the unknown term on the right of 5.47, the divergence theorem is used as follows

5.52 $$\iint_D \psi J(\psi)\,dv = -\iint_D \psi_{,i} a^{ij} \psi_{,j}\,dv + \oint_B \psi \frac{\partial\psi}{\partial\nu} ds - \iint_D q\,\psi^2 dv$$

$$\leq -\iint_D \psi_{,i} a^{ij} \psi_{,j}\,dv + \oint_B H \frac{\partial\psi}{\partial\nu} ds - \sum_{I=1}^{d} \varepsilon^I \tilde{m}_I \left(\frac{2}{N-2}\right) \oint_B (x^i - x^i_I) r_I^{-2} n_i H^2 ds$$

$$- \bar{m} \iint_D a^{ij} \psi_{,i} \psi_{,j}\,dv$$

where we have use of an appropriate form of 4.40. Transposition of 5.52 yields

5.53 $$\left(1 - \bar{m} - \frac{2\alpha_1 R^2}{N^2 a_0}\right) \iiint_D a^{ij} \psi_{,i} \psi_{,j}\,dv$$

$$\leq -\sum_{I=1}^{d} \varepsilon^I \tilde{m}_I \left(\frac{2}{N-2}\right) \oint_B (x^i - x^i_I) r_I^{-2} n_i H^2 ds + \frac{1}{2\alpha_1} \iint_D F^2 dv$$

$$+ \frac{\alpha_2}{2} \oint_B \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds + \frac{1}{2\alpha_2} \oint_B H^2 ds + \frac{\alpha_1}{N} \oint_B (x^i - \underline{x}^i) n_i H^2 ds$$

for any positive numbers α_1 and α_2. Since we are seeking a bound for the integral of the square of the conormal derivative over B we are led to choose the arbitrary constants in the last inequality as follows

5.54 $$\alpha_1 = (1-\bar{m}) \frac{N^2 a_0}{4\underline{R}^2}$$

and

5.55 $$\alpha_2 = \frac{1}{4}\bar{m}_3(1-\bar{m})\ (\bar{c}_1 + \frac{2\bar{M}_1}{a_o} + \frac{4\bar{M}_{14}\underline{R}^2}{N^2 a_o})^{-1} .$$

Substitution of these results into 5.47 yields

5.56 $$\oint_B \left(\frac{\partial\psi}{\partial\nu}\right)^2 ds \leq \mathcal{B}$$

where $\mathcal{B}$ is an appropriate computable constant which is such that

5.57 $$\mathcal{B} = 0$$

if

5.58 $$F = H = 0$$

For N = 2 or N = 3 we use the results of pages 562 and 563 of [21] to estimate the unknown volume term in 5.44. These results are directly applicable since 5.53 provides a bound for

5.59 $$\iint_D \sum_1^N \psi_{,i}^2\, dv .$$

If $N > 3$ then $\mathcal{F}$ is defined by (4.18) of [21] so that

5.60 $$-\mathcal{F} \leq M \quad , \quad J^*(\mathcal{F}) < 0 \qquad\qquad x \in D$$

where

5.61 $$J^*(\mathcal{F}) = (a^{ij}\mathcal{F}_{,j})_{,i} - 2q\mathcal{F} .$$

The divergence theorem may be used to establish that

5.62
$$\iint_D \{ 2\mathcal{F}\psi\, J(\psi) - \psi^2 J^*(\mathcal{F}) \} \, dV$$
$$\leq 2M\mathcal{D} + 2\{\mathcal{B}\}^{1/2} \{ \oint_B \mathcal{F}^2 H^2 \, ds \}^{1/2}$$
$$- \oint_B H^2 \frac{\partial \mathcal{F}}{\partial \nu} \, ds$$

and, since the right side of this equation is computable, it is evident that the desired pointwise bound is available using the techniques and the results of the paper by Payne and Weinberger. In the above equation $\mathcal{B}$ is given by 5.56 while $\mathcal{D}$ is the bound for the integral of $a^{ij} \psi_{,i} \psi_{,j}$ which is obtained from 5.53 after application of 5.56.

CHAPTER VI

EXAMPLE

1. Explicit Non-Linear Mixed Problem

In this section a bound shall be obtained for the solution of a non-linear boundary value problem which is of the type considered in the first section of Chapter III.

We suppose that V is a three dimensional domain bounded by the hyperplanes y = o and y = 1 and by the surface whose equation is

6.1 $$x^{1^2} + x^{2^2} = 1.$$

The boundary value problem is explicitly stated as follows:

6.2 $$J(W) = \frac{\partial^2 W}{\partial x^{1^2}} + \frac{\partial^2 W}{\partial x^{2^2}} - W - \frac{\partial W}{\partial y} = (A + |W|)^{-1} - \tfrac{1}{2} r^2 + r \qquad (x,y) \in V, \ -\infty < W < \infty$$

$$W(x,0) = 0 \quad (x,y) \in D(0) \ ; \quad \frac{\partial W}{\partial \nu} = 1 \quad (x,y) \in S$$

where

6.3 $$A > o.$$

From the above we obtain

$$6.4 \qquad \begin{cases} \tilde{M}_1 = \frac{1}{A^2} & m_1 = 1 \\ \tilde{M}^1 = 0 & m_2 = 1 \\ \tilde{M}^2 = 0 & a_0 = 1 \\ \tilde{M}_2 = 0 & a_1 = 1 \end{cases}$$

By inspection one observes that the differential operator of 6.2 is slightly different from that of 3.1. The remarks which are found at the conclusion of Chapter III suggest the reason for this difference. The additional term is combined with the differential operator in order to indicate the improvement which follows from such an association. If this were not done, then

$$6.4\text{ a} \qquad \tilde{M}_1 = \frac{1}{A^2} + 1$$

and by close inspection of the results which are obtained below, it is evident that the sharpness of the error bound is materially diminished.

Following the techniques of the earlier chapters we let

$$6.5 \qquad \varphi = \frac{1}{2} r^2$$

and then calculate

$$6.6 \qquad \begin{cases} F = (A + \frac{1}{2} r^2)^{-1} \\ G = -\frac{1}{2} r^2 \\ L = 0 \end{cases}$$

Even though φ satisfies the boundary conditions on S and for large A approximates the solution of the differential equation, it is evident that φ does not closely coincide with w throughout V since on D(o) the approximation is very poor.

The set of auxiliary functions which were introduced in the remarks accompanying equation 3.23 may be defined as follows:

$$6.7 \qquad f^i = x^i \qquad\qquad i = 1 \text{ and } 2$$

Thus we have

$$6.8 \qquad \begin{cases} m_4 = 1 & M_3 = 0 \\ M_1 = 1 & M_4 = 0 \\ M_2 = 2 & M_5 = 0. \end{cases}$$

In what follows the solution of 6.2 is bounded at the point p = (0,1). From 3.16 we have

$$6.9 \qquad |w(0,1) - \chi(0,1)| \le \left| \lim_{y \to 1} \iiint_{V(y)} \psi J^*(\gamma_p) dv \right|$$

$$+ \left| \lim_{y \to 1} \iiint_{V(y)} \gamma_p \{ f(x,y,w,\nabla w) - f(x,y,\varphi,\nabla\varphi) \} dv \right|$$

$$+ \left\{ \iint_S \psi^2 ds \right\}^{1/2} \left\{ \iint_S \left(\frac{\partial \gamma_p}{\partial \nu} \right)^2 ds \right\}^{1/2}$$

where

6.10 $$\gamma_p = \left[4\pi(1-y)\right]^{-1} \exp\left[-r^2 \left\langle 4(1-y)\right\rangle^{-1}\right]$$

and

6.11 $$\mathfrak{J}(\gamma_p) = \frac{\partial^2 \gamma_p}{\partial x'^2} + \frac{\partial^2 \gamma_p}{\partial x^{2\,2}} - \gamma_p + \frac{\partial \gamma_p}{\partial y}$$

Since φ is such that L is zero on S, we find it advantageous to choose the arbitrary constants of 3.35 in a slightly different manner than in equations 3.38 to 3.42. Thus by setting

6.12 $$\alpha_2 = \infty$$

and

6.13 $$\alpha_4 = 0,$$

equation 3.35 may be rewritten as

6.14 $$\iiint\limits_V \sum_1^2 \tilde{\varphi}_{,i}^2 \, dV + \left(K + 1 - \frac{1}{A^2} - \frac{\alpha_3}{2}\right) \iiint\limits_V \tilde{\varphi}^2 \, dV$$

$$\leq \frac{1}{4} \exp(2K) \iint\limits_{D(0)} r^4 \, dS$$

$$+ \frac{1}{2\alpha_3} \iiint\limits_V \left(A + \tfrac{1}{2} r^2\right)^{-2} \exp\left[2K(1-y)\right] dV .$$

Substitution of this into 3.24 yields

6.15 $$\iint_S \tilde{\varphi}^2 ds \le \frac{\alpha_1 \pi}{12} \exp(2K) + \frac{\alpha_1 \pi}{2A(2A+1)\alpha_3 K}\{-1+\exp(2K)\}$$

$$+\left(-\alpha_1 K - \alpha_1 + \frac{\alpha_1}{A^2} + \frac{\alpha_3 \alpha_1}{2} + \frac{1}{\alpha_1} + 2\right) \iiint_V \tilde{\varphi}^2 dv.$$

Due to the form of the above we set

6.16 $$K = -1 + \frac{1}{A^2} + \frac{\alpha_3}{2} + \frac{1}{\alpha_1^2} + \frac{2}{\alpha_1}$$

and then seek to determine the positive constants α_1 and α_3 so that the right side of 6.15 is minimized. To solve such a minimum problem is extremely difficult; hence in practice we ordinarily choose the arbitrary constants in some admissible manner and then seek to improve our result by a better choice of the approximation function φ. For example we assume that

6.17 $$\frac{1}{A^2} < \frac{7}{16}$$

and set

6.18 $$\alpha_1 = 4$$

and

6.19 $$\alpha_3 = 2\left(1 - \frac{1}{A^2} - \frac{1}{16} - \frac{1}{2}\right) = \frac{7}{8} - \frac{2}{A^2} .$$

From the above and 6.16 it follows that

6.20 $$K = 0.$$

Equation 6.15 may now be rewritten as

6.21 $$\iint_S \psi^2 ds \leq \frac{\pi}{3} + \frac{4\pi}{A(2A+1)\left(\frac{7}{8} - \frac{2}{A^2}\right)} = \mathcal{B}.$$

We use 6.10 to obtain

6.22 $$\left\{\iint_S \left(\frac{\partial \gamma_P}{\partial \nu}\right)^2 ds\right\}^{1/2} = \{.157\}^{1/2} = .384$$

and hence

6.23 $$\left\{\iint_S \psi^2 ds\right\}^{1/2} \left\{\iint_S \left(\frac{\partial \gamma_P}{\partial \nu}\right)^2 ds\right\}^{1/2}$$

$$\leq .384\, \mathcal{B}^{1/2}$$

where $\mathcal{B}$ is given by 6.21.

To bound the first term on the right of 6.9 we introduce an auxiliary function $\mathcal{F}$ which is such that the conclusions of Theorem 2.2 are valid. The auxiliary function used in this example is not identical to that defined by 2.39. While the original function could be used for the present problem, advantage is made of the simplicity of the differential operator to obtain a function which lends itself to the integrations which are required for the determination of the desired bound. Let

6.24 $$\mathcal{F} = (1-y)^{-\frac{1}{2}} \exp\left[-r^2 \langle 4(1-y)\rangle^{-1}\right].$$

Clearly

6.25 $\mathcal{F} \geq 0$ for $(x, y) \in V$

and since

6.26
$$\bar{J}(\mathcal{F}) = \frac{\partial^2 \mathcal{F}}{\partial x^{1^2}} + \frac{\partial^2 \mathcal{F}}{\partial x^{2^2}} + \frac{\partial \mathcal{F}}{\partial y}$$

$$= -\frac{1}{2}(1-y)^{-\frac{3}{2}} \exp\left[r^2 \langle 4(1-y) \rangle^{-1}\right] \leq 0$$

for $(x,y) \in V$, it follows that the conclusions of Theorem 2.2 are valid as was indicated above. For the present differential operator equation 3.57 becomes

6.27
$$\lim_{\substack{y \to 1 \\ y < 1}} \iiint_{V(y)} \left\{ 2\mathcal{F}\, \tilde{\varphi}\, \tilde{J}(\tilde{\varphi}) - \tilde{\varphi}^2 \bar{J}(\mathcal{F}) \right.$$

$$\left. + 2\mathcal{F} \sum_1^2 \tilde{\varphi}_{,i}^2 + 2(K+1)\mathcal{F}\tilde{\varphi}^2 \right\} dV$$

$$= -\iint_S \tilde{\varphi}^2 \frac{\partial \mathcal{F}}{\partial \nu} ds + \iint_{D(0)} \mathcal{F} G^2 \exp(2K)\, ds.$$

From 3.62 and the above we have

6.28 $K = K_2 \geq (-1 + \frac{1}{A^2}) < 0$

where the strict inequality holds on the right due to 6.17. In what follows we set

6.29 $K_2 = 0$

to simplify the numerical calculations. (If greater accuracy were desired, one would of course take K_2 equal to $A^{-2}-1$). From equations 3.57 to 3.65 we obtain

6.30
$$\iiint_V \{2\mathcal{F}\psi F - \psi^2 \bar{J}(\mathcal{F})\}\, dV$$

$$\leq \mathcal{B} \max_S \left\{-\frac{\partial \mathcal{F}}{\partial \nu}\right\} + .217 = \mathcal{J}_2$$

To use 3.69 it is also necessary to compute

6.31
$$\iiint_V [-\bar{J}(\mathcal{F})]^{-1} (\mathcal{F} F)^2 \exp[2K_2(1-y)]\, dV$$

$$= 2\iiint_V (1-y)^{1/2} (A + \tfrac{1}{2}r^2)^{-2} \exp[2K_2(1-y) - r^2\langle 4(1-y)\rangle^{-1}]\, dV$$

$$\leq 2\iiint_V (1-y)^{1/2} A^{-2} \exp[-r^2\langle 4(1-y)\rangle^{-1}]\, dV$$

$$= 3.24\, A^{-2}$$

and

6.32
$$\iiint_V [-\bar{J}(\mathcal{F})]^{-1}\, J^*(\gamma_p)^2\, dV$$

$$= (4\pi)^{-2} \iiint_V 2(1-y)^{-\frac{1}{2}} \exp[-r^2\langle 4(1-y)\rangle^{-1}]\, dV$$

$$= 0.0418$$

By combining the foregoing and equation 3.69, it follows that

6.33
$$\left|\lim_{\substack{y\to 1\\ y<1}} \iiint_{V(y)} \psi\, J^*(\gamma_p)\, dV\right|$$

$$\leq 0.204\left\langle 1.8A^{-1} + \left\{+.217 + 1.64\left[\frac{\pi}{3} + \frac{4\pi}{A(2A+1)\left(\frac{7}{8}-\frac{2}{A^2}\right)}\right] + 3.24A^{-2}\right\}^{1/2}\right\rangle.$$

To bound the second term on the right of 6.9, we could use the techniques associated with equations 3.69 to 3.75; however, since

6.34
$$J^*(\gamma_p) = -\gamma_p,$$

we apply Schwarz's inequality to the expression under consideration and write

6.35
$$\left|\lim_{\substack{y\to 1\\ y<1}} \iiint_{V(y)} \gamma_p \{f(x,y,w,\nabla w) - f(x,y,\varphi,\nabla\varphi)\} dV\right|$$

$$\leq \frac{1}{A} \iiint_V \gamma_p |\psi|\, dV$$

$$\leq \frac{1}{A}\left\{\iiint_V [-\bar{J}(\mp)]^{-1} J^*(\gamma_p)^2 dV\right\}^{1/2}\left\{\iiint_V [-\bar{J}(\mp)]\,\psi^2 dV\right\}^{1/2}$$

$$\leq 0.204\, A^{-1}\left\langle 1.8A^{-1} + \left\{1.64\left[\frac{\pi}{3} + \frac{4\pi}{A(2A+1)\left(\frac{7}{8}-\frac{2}{A^2}\right)}\right]\right.\right.$$

$$\left.\left. + 3.24\, A^{-2} + .217\right\}^{1/2}\right\rangle.$$

The bound for the solution is completed by calculating the function $\chi(p)$ which is given by 3.17. One easily obtains

6.36 $$\varphi(0,1) + \iint_S L\gamma_p \, ds = 0$$

6.37 $$\iint_{D(0)} G\gamma_p \, ds = -0.06$$

and

6.38 $$-.2212 A^{-2} \le -\iiint_V \gamma_p F \, dv \le -.2212 (A + \tfrac{1}{2})^{-2}.$$

From 6.9 and the above we write as the desired bound for the solution

6.39 $$-0.384 \left\{ \frac{\pi}{3} + \frac{4\pi}{A(2A+1)\left(\frac{7}{8} - \frac{2}{A^2}\right)} \right\}^{1/2} - 0.204 \left(1 + A^{-1}\right) \Big\langle 1.8 A^{-1}$$

$$+ \left\{ +.217 + 1.64 \left[\frac{\pi}{3} + \frac{4\pi}{A(2A+1)\left(\frac{7}{8} - \frac{2}{A^2}\right)} \right] + 3.24 A^{-2} \right\}^{1/2} \Big\rangle$$

$$+ 0.06 + .2212 \left(A + \tfrac{1}{2}\right)^{-2}$$

$$\le w(0,1)$$

$$\le +0.384 \left\{ \frac{\pi}{3} + \frac{4\pi}{A(2A+1)\left(\frac{7}{8} - \frac{2}{A^2}\right)} \right\}^{1/2} + 0.204 \left(1 + A^{-1}\right) \Big\langle 1.8 A^{-1}$$

$$+ \left\{ +.217 + 1.64 \left[\frac{\pi}{3} + \frac{4\pi}{A(2A+1)\left(\frac{7}{8} - \frac{2}{A^2}\right)} \right] + 3.24 A^{-2} \right\}^{1/2} \Big\rangle$$

$$+ 0.06 + .2212 A^{-2}.$$

For example let us suppose that

6.40 $$A = 4$$

then one obtains from 6.42 the following result

6.41 $$-0.95 \leqslant w(0,1) \leqslant +1.09$$

When bounds for the solution of more complicated problems are desired, the techniques are exactly the same as used above; however the integrals which result may be quite difficult to evaluate.

SELECTED BIBLIOGRAPHY

1. D. Bernstein, Existence Theorems in Partial Differential Equations, Princeton, Princeton University Press, 1950.

2. R. Courant and D. Hilbert, Methoden der Mathematischen Physik, Vol. II, Berlin, Springer, 1937.

3. F. Dressel, The fundamental solution of the parabolic equation, Duke Math. J., 7 (1940), 186-203.

4. ______, The fundamental solution of the parabolic equation, Duke Math. J., 13 (1946), 61-70.

5. G. Evans, The non-homogeneous differential equation of the parabolic type, Am. J. of Math., 37 (1915), 431-438.

6. G. Fichera, Formule di maggiorazione connesse ad una classe di transformazioni lineari, Annali di Mat. Pura e Appl., 36 (1954), 273-296.

7. ______, Methods of functional linear analysis in mathematical physics, Proc. of the Int. Cong. of Math. 1954, Amsterdam, North-Holland Pub. Co. (1956), 216-228.

8. ______, On a unified theory of boundary value problems for elliptic-parabolic equations of second order, MRC Technical Summary Report #110, Madison, University of Wisconsin (1959).

9. A. Friedman, Boundary estimates for second order parabolic equations and their applications, J. Math. Mech., 7 (1958), 771-792.

10. ______, On quasi-linear parabolic equations of second order, J. Math. Mech., 7 (1958), 793-809.

11. ______, Heat transfer between solids and gases under non-linear boundary conditions, Technical Report #27, Dept. of Math., University of California (1958).

12. ______, On quasi-linear parabolic equations of second order, J. Math. Mech., 9 (1960), 539-556.

13. M. Gevrey, Sur les équations aux dérivées partielles du type parabolique, J. de Math., 9 (1913), 305-471.

14. ______, Sur la nature analytique des solutions des équations aux dérivées partielles, Ann. Ec. Norm., 3 (1918), 129-190.

15. E. Goursat, Cours d'Analyse Mathematique, Paris, Gauthier-Villars, 1942.

16. L. Hörmander, Uniqueness theorems and estimates for normally hyperbolic partial differential equations of the second order, Comptes Rendus du Douzième Congrès des Mathèmaticiens Scandinaves Tenu à Lund 10-15 Août 1953, Lund, Hakan Ohlssons Boktryckeri (1954), 105-115.

17. B. Hubbard, Bounds for the Eigenvalues of the Free and Fixed Membranes by Finite Difference Methods, Unpublished doctoral dissertation, The University of Maryland, College Park, 1960.

18. O. Kellogg, Foundations of Potential Theory, Berlin, Springer, 1929.

19. H. Lieberstein, A continuous method in numerical analysis applied to examples from a new class of boundary value problems, MRC Technical Summary Report #175, Madison, University of Wisconsin (1960).

20. J. Nash, Parabolic equations, Proc. of the Nat. Acad. of Sci., 43 (1957), 754-758.

21. L. Payne and H. Weinberger, New bounds for solutions of second order elliptic partial differential equations, Pac. J. Math., 8 (1958), 551-573.

22. A. Piskorek, Sur certain problemes aux limites pour l'équation semi-lineaire parabolique normal, Polon. Sci. Ser. Sci. Math. Astro. Phys. 6 (1958), 505-510.

23. W. Pogorzelski, Problème aux limites aux dérivées tangentielles pour l'équation parabolique, Ann. Ec. Norm. Sup., 75 (1958), 19-35.

24. S. Robinson, Some properties of the fundamental solution of the parabolic equation, Duke Math. J., 27 (1960), 195-219.

25. E. Rothe, Über die grundlosung bei parabolischen gleichungen, Math. Zeitschrift, 33 (1931), 488-504.

26. J. Schauder, Der fixpunktsatz in funktionalraümen, Studia Math., 2 (1930), 171-180.

27. P. Zeragiya, Boundary problems for certain non-linear equations of the parabolic type, Akad. Nauk. Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze, 24 (1957), 195-221 (Russian).

28. ______, Application of the čaplygin method to obtaining an approximate solution of a non-linear equation of parabolic type, Soobšč. Akad. Nauk. Gruzin, SSR., 18 (1957), 647-654 (Russian).